



Parametric Constructions of Monogenic and Exceptional Sextic Fields

Aruna C¹ and Parthiban V*,¹

ABSTRACT

We investigate two parametric families of irreducible sextic polynomials over \mathbf{Q} , denoted $g(x; t)$ and $h(x; t)$, and the number fields they generate. For integers t in suitable ranges, we show that the fields $K_t = \mathbf{Q}(\alpha)$ and $L_t = \mathbf{Q}(\theta)$ — where α and θ are roots of $g(x; t)$ and $h(x; t)$, respectively — exhibit rich arithmetic and algebraic structure.

In particular, both families define exceptional number fields, and we prove that for infinitely many t , the fields are monogenic. We also show that K_t contains real quadratic subfields of the form $\mathbf{Q}(\sqrt{t^2 - 4})$, and that every real quadratic field embeds in some K_t . Meanwhile, each L_t contains a cubic subfield of the form $\mathbf{Q}(\theta^2 - \theta)$. These results suggest that exceptional and monogenic number fields arise naturally and frequently in well-structured parametric families.

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1 Introduction

The exploration of algebraic number fields of small degree remains a fundamental aspect of understanding the arithmetic of field extensions. Among these, *sextic fields*, degree six extensions of the rational numbers, occupy an interesting position: they are complex

*Corresponding author

. VIT University, Chennai, India.

aruna.2020@vitstudent.ac.in; parthiban.v@vit.ac.in

enough to exhibit rich behavior, yet remain accessible to explicit investigation. A central question in this context is that of *monogeneity*—whether a single element generates the ring of integers of a number field, thereby forming a *power integral basis*. While monogenic fields are relatively common in lower degrees, they become increasingly rare in higher-degree extensions.

In this article, we examine a parametric family of sextic fields

$$K_t = \mathbf{Q}(\alpha),$$

defined by the irreducible polynomial $g(x; t)$, where $t \geq 8$ is an integer and α is a root of g . We show that these fields are *monogenic for infinitely many values of t* , and that they exhibit structural properties not typically observed in known families. Notably, each K_t contains the real quadratic subfield

$$\mathbf{Q}(\sqrt{t^2 - 4}),$$

and, even more strikingly, *every* real quadratic field appears as a subfield of some K_t , demonstrating the versatility of these constructions.

In addition to this first family, we introduce a second parametric family of sextic fields defined by another polynomial $h(x; t)$. This family likewise displays strong structural features, including monogeneity and the presence of exceptional units, and it contains real cubic subfields of the form

$$\mathbf{Q}(\theta^2 - \theta).$$

This study extends previous work on quartic fields [1], which was cited by Washington and Pincus [2] in their investigation of the simplest quartic fields. That earlier work provided an alternate viewpoint on the field-isomorphism problem through the use of exceptional units. In parallel, the present article demonstrates that the parametric families of sextic fields considered here contain infinitely many non-isomorphic fields, offering new insight into the structure of degree-six extensions.

These results deepen our understanding of the relationship between monogeneity and exceptional field structure in sextic number fields. The families introduced here furnish new examples of *monogenic number fields with additional algebraic structure*, expanding the catalog of such fields and opening further avenues for research in the arithmetic of number fields.

2 Preliminaries

Definition 2.1. *Let F be a number field. A unit $\lambda \in \mathcal{O}_F$ is called an **exceptional unit** if both λ and $1 - \lambda$ are units in \mathcal{O}_F . A number field F is said to be **exceptional** if its ring of integers contains at least one such unit.*

The arithmetic of units in number fields has been investigated extensively by Nagell in a sequence of papers [4, 5, 6, 7, 8]. The following elementary criterion, due to Nagell, gives a convenient way to detect exceptional units arising from the roots of a polynomial.

Lemma 2.2 ([7]). *Let $f(x) \in \mathbb{Z}[x]$ be monic and irreducible. If $|f(0)| = 1$ and $|f(1)| = 1$, then every root α of f is an exceptional unit in the number field $\mathbf{Q}(\alpha)$.*

Definition 2.3 (Sturm sequence). *Let $f(x) \in \mathbb{R}[x]$ be a nonconstant polynomial without multiple roots. The Sturm sequence associated with f is the finite sequence of polynomials*

$$r_0(x), r_1(x), \dots, r_s(x),$$

defined recursively by

$$\begin{aligned} r_0(x) &= f(x), \\ r_1(x) &= f'(x), \\ r_{k+1}(x) &= -\operatorname{rem}(r_{k-1}(x), r_k(x)), \quad k \geq 1, \end{aligned}$$

where $\operatorname{rem}(g, h)$ denotes the remainder in the Euclidean division of $g(x)$ by $h(x)$. The construction terminates when $r_s(x)$ is a nonzero constant polynomial.

Theorem 2.4 (Sturm's theorem). *Let $f(x) \in \mathbb{R}[x]$ be a nonconstant polynomial with no repeated real roots, and let*

$$r_0(x), r_1(x), \dots, r_s(x)$$

be the Sturm sequence associated with f . For $t \in \mathbb{R}$, denote by $V(t)$ the number of sign changes in the sequence

$$(r_0(t), r_1(t), \dots, r_s(t)),$$

where zero values are omitted.

Then the total number of distinct real zeros of $f(x)$ is given by

$$V(-\infty) - V(+\infty),$$

where $V(\pm\infty)$ denotes the number of sign changes obtained from the signs of the leading terms of $r_0(x), \dots, r_s(x)$ as $x \rightarrow \pm\infty$.

Definition 2.5. A number field K/\mathbf{Q} is called **monogenic** if its ring of integers can be written as a power basis, that is,

$$\mathcal{O}_K = \mathbb{Z}[\alpha] = \mathbb{Z} + \alpha\mathbb{Z} + \dots + \alpha^n\mathbb{Z}$$

for some element $\alpha \in K$, where $n = [K : \mathbf{Q}] - 1$.

Definition 2.6. Let $f(x; t) \in \mathbb{Z}[x]$ be a monic irreducible polynomial depending on a parameter t . Denote by $\Delta(t) \in \mathbb{Z}[t]$ the discriminant of f with respect to x . Suppose

$$\Delta(t) = \prod_{i=1}^r p_i(t)^{e_i},$$

where the $p_i(t)$ are pairwise distinct irreducible polynomials in $\mathbb{Z}[t]$. The **reduced discriminant** of f is then defined as

$$\Delta^{\text{red}}(t) := \prod_{i=1}^r p_i(t).$$

König's theorem provides a powerful tool for establishing monogenicity for parameterized families of number fields. We recall it here for later use.

Theorem 2.7 (König, 2018 [3]). *Let*

$$h(t, x) = h_1(x) - th_2(x) \in \mathbb{Z}[t, x]$$

be a polynomial with Galois group G_h over $\mathbf{Q}(t)$, and let $\Delta^{\text{red}}(t)$ be its reduced discriminant. Suppose the following two conditions hold:

- (i) $\Delta^{\text{red}}(t)$ has no irreducible factor of degree at least 4;
- (ii) no prime number p divides all integer values taken by $\Delta^{\text{red}}(t)$.

Then there exist infinitely many integers t_0 such that the field $\mathbf{Q}(\alpha)$, where α is a root of $h(t_0, x) = 0$, is monogenic with Galois group G_h . More precisely, the set

$$\{\alpha^i : i = 0, \dots, \deg(h) - 1\}$$

forms a power integral basis for the ring of integers of $\mathbf{Q}(\alpha)$.

3 Sextic Families

3.1 The number field K_t with quadratic subfield

Theorem 3.1. *Let $t \geq 8$ be an integer and consider the following family of polynomials,*

$$g(x; t) = x^6 - 2x^5 + x^4 + tx^3 - tx^2 + 1$$

each of which is irreducible over \mathbf{Q} . Let α be a root of $g(x; t)$, and define the number field $K_t = \mathbf{Q}(\alpha)$.

Then we have

- (i) K_t is an exceptional number field of unit rank 4.
- (ii) The real quadratic field $\mathbf{Q}(\sqrt{t^2 - 4})$ is a subfield of K_t .
- (iii) Every real quadratic field $\mathbf{Q}(\sqrt{d})$ is contained as a subfield of some K_t .
- (iv) The number field $\mathbf{Q}(\alpha)$ is monogenic for infinitely many $t \in \mathbb{Z}$.

Proof. We will now prove the statements one by one, from (i) to (iv).

Proving irreducibility is straightforward, since the polynomial is reducible only for a small number of integer values of t , all of which are excluded by assuming $t \geq 8$.

Proof of (i). We have $g(0) = 1$ and $g(1) = -1$. By Lemma 2.2, this implies that any root α of the polynomial is an exceptional unit. Therefore, the number field $\mathbf{Q}(\alpha)$ generated by

α is an exceptional number field. In particular, the element $1 - \alpha$ is also a unit in this field.

We calculate sturm sequence to prove that the unit rank is 4. The Sturm sequence for the polynomial $r_0(x)$ (with parameter t) is

$$\begin{aligned} r_0(x) &= x^6 - 2x^5 + x^4 + tx^3 - tx^2 + 1, \\ r_1(x) &= 6x^5 - 10x^4 + 4x^3 + 3tx^2 - 2tx, \\ r_2(x) &= \frac{2}{9}x^4 + \left(-\frac{1}{2}t - \frac{2}{9}\right)x^3 + \frac{1}{2}tx^2 + \frac{1}{9}tx - 1, \\ r_3(x) &= \left(-\frac{243}{8}t^2 + 9t\right)x^3 + \left(\frac{243}{8}t^2 - 9t\right)x^2 + \left(\frac{27}{4}t^2 - 27\right)x - \frac{243}{4}t + 18, \\ r_4(x) &= \left(-\frac{4}{81}t^2 + \frac{16}{81}\right)x^2 + \left(\frac{8}{243}t^2 - \frac{32}{243}\right)x, \\ r_5(x) &= \left(-\frac{27}{2}t^2 + 2t + 27\right)x + \frac{243}{4}t - 18, \\ r_6(x) &= \frac{-\frac{4}{27}t^3 + \frac{745}{729}t^2 + \frac{16}{27}t - \frac{2980}{729}}{t^4 - \frac{8}{27}t^3 - \frac{2900}{729}t^2 + \frac{16}{27}t + 4}, \end{aligned}$$

Evaluating the signs of r_0, \dots, r_6 at $\pm\infty$ yields the value of $V(-\infty) - V(+\infty) = 4$ from which the number of real roots of is 4; the corresponding counts of real and complex embeddings then give the unit rank through the formula is 4.

Proof of (ii). Let α be a root of the polynomial $g(x; t)$. Then α satisfies equation

$$\alpha^6 - 2\alpha^5 + \alpha^4 + t\alpha^3 - t\alpha^2 + 1 = 0.$$

Define $\beta = \alpha^3 - \alpha^2$. Substituting into the expression, we find that

$$t^2 - 4 = \frac{(\beta^2 - 1)^2}{\beta^2}.$$

This equation implies that $t^2 - 4$ is a square in the field $K_t = \mathbf{Q}(\alpha)$, the field obtained by adjoining α to $\mathbf{Q}(t)$. Consequently, K_t contains $\sqrt{t^2 - 4}$, and hence it contains the real quadratic field $\mathbf{Q}(\sqrt{t^2 - 4})$.

Proof of (iii). Now, let d be a positive square-free integer and let u be a unit in the real quadratic field $\mathbf{Q}(\sqrt{d})$. Suppose $2u = t + s\sqrt{d}$ for some integers $t, s \in \mathbb{Z}$. Calculating the norm of u , we obtain:

$$\text{Norm}(u) = u \cdot \bar{u} = 1 \quad \Rightarrow \quad (t + s\sqrt{d})(t - s\sqrt{d}) = 4,$$

Which gives:

$$t^2 - ds^2 = 4 \quad \Rightarrow \quad t^2 - 4 = ds^2.$$

This shows that both \sqrt{d} and $\sqrt{t^2 - 4}$ generate the same quadratic field. That is, they are two distinct primitive elements for the same real quadratic extension of \mathbf{Q} . Therefore,

$\mathbf{Q}(\sqrt{d})$ is a subfield of the exceptional sextic field K_t .

Proof of (iv). Computation of the discriminant. Let

$$g(x) = x^6 - 2x^5 + x^4 + tx^3 - tx^2 + 1.$$

We first compute the derivative:

$$g'(x) = 6x^5 - 10x^4 + 4x^3 + 3tx^2 - 2tx.$$

Since $\deg g = 6$, the discriminant is

$$\Delta(g) = (-1)^{\frac{6 \cdot 5}{2}} \operatorname{Res}(g, g') = (-1)^{15} \operatorname{Res}(g, g') = -\operatorname{Res}(g, g').$$

The resultant $\operatorname{Res}(g, g')$ is the determinant of the 11×11 Sylvester matrix $S(g, g')$, constructed from the coefficients of g and g' .

Thus,

$$\operatorname{Res}(g, g') = \det S(g, g').$$

Evaluating this determinant and simplifying in $\mathbb{Z}[t]$ gives

$$\operatorname{Res}(g, g') = (108t - 745)(t - 2)^3(t + 2)^3.$$

Since $\Delta(g) = -\operatorname{Res}(g, g')$, we obtain

$$\Delta(g) = -(108t - 745)(t - 2)^3(t + 2)^3.$$

The corresponding reduced discriminant is

$$\Delta^{\operatorname{red}}(t) = (-1)(108t - 745)(t - 2)(t + 2),$$

which has no irreducible factor of degree ≥ 4 .

Evaluating $\Delta^{\operatorname{red}}(t)$ at two distinct integers:

$$\Delta^{\operatorname{red}}(1) = -1 \times (108 \cdot 1 - 745)(1 - 2)(1 + 2) = -1911 = -1 \times 3^3 \times 7^2 \times 13,$$

$$\Delta^{\operatorname{red}}(3) = -1 \times (108 \cdot 3 - 745)(3 - 2)(3 + 2) = 2105 = 5^3 \times 421.$$

Since these two values have no common prime factors, there exists no prime p dividing $\Delta^{\operatorname{red}}(t)$ for all integers t . Therefore, by Theorem 2.7, we conclude that there are infinitely many values of t for which the number field $\mathbf{Q}(\alpha)$, with α a root of $g(x; t)$, is monogenic. \square

Similar to the previously discussed family of sextic polynomials, we now exhibit a different infinite family—still rare—consisting of irreducible polynomials that share analogous arithmetic and algebraic properties.

3.2 Number field L_t with cubic subfield

Theorem 3.2. *Let $t \geq 3$ be an integer and consider the following family of polynomials,*

$$h(x; t) = x^6 - 3x^5 + 3x^4 - x^3 - tx^2 + tx + 1$$

each of which is irreducible over \mathbf{Q} . Let θ be a root of $h(x; t)$, and define the number field $L_t = \mathbf{Q}(\theta)$. Then we have

- (i) L_t is an exceptional number field of unit rank 4.
- (ii) L_t contains the real cubic field $\mathbf{Q}(\theta^2 - \theta)$.
- (iii) The number field $\mathbf{Q}(\theta)$ is monogenic for infinitely many $t \in \mathbb{Z}$.

Proof. The irreducibility of the polynomial can be easily verified, as it is reducible only for a limited set of integer values of t , all of which are excluded by the assumption $t \geq 3$.

Proof of (i). Since $h(0) = 1$ and $h(1) = 1$, it follows from Lemma 2.2 that any root θ of $h(x)$ is an exceptional unit. Therefore, the number field $L_t = \mathbf{Q}(\theta)$ is an exceptional number field. To determine the unit rank of the degree-6 number field generated by a root of $r_0(x)$, we apply Sturm's theorem. The Sturm sequence associated with $r_0(x)$ (with parameter t) is given by

$$\begin{aligned} r_0(x) &= x^6 - 3x^5 + 3x^4 - x^3 - tx^2 + tx + 1, \\ r_1(x) &= 6x^5 - 15x^4 + 12x^3 - 3x^2 - 2tx + t, \\ r_2(x) &= \frac{1}{4}x^4 - \frac{1}{2}x^3 + \left(\frac{2}{3}t + \frac{1}{4}\right)x^2 - \frac{2}{3}tx - \frac{1}{12}t - 1, \\ r_3(x) &= 16tx^3 - 24tx^2 + (8t - 24)x + 12, \\ r_4(x) &= \left(\frac{-\frac{2}{3}t^2 + \frac{1}{16}t - \frac{3}{8}}{t}\right)x^2 + \left(\frac{\frac{2}{3}t^2 - \frac{1}{16}t + \frac{3}{8}}{t}\right)x + \frac{\frac{1}{12}t^2 + t - \frac{3}{32}}{t}, \\ r_5(x) &= \left(\frac{-2t^3 + \frac{27}{2}}{t^2 - \frac{3}{32}t + \frac{9}{16}}\right)x + \frac{t^3 - \frac{27}{4}}{t^2 - \frac{3}{32}t + \frac{9}{16}}, \\ r_6(x) &= -\frac{1}{4}t - \frac{63}{64}. \end{aligned}$$

Evaluating the signs of r_0, \dots, r_6 at $\pm\infty$ yields $V(-\infty) - V(+\infty) = 4$, which shows that $r_0(x)$ has exactly four real roots. Hence the associated number field has 4 real embeddings and one pair of complex embeddings, implying that the unit rank is 4.

Proof of (ii). Since θ is a root of the polynomial $h(x)$, we have

$$\theta^6 - 3\theta^5 + 3\theta^4 - \theta^3 - t\theta^2 + t\theta + 1 = 0.$$

This expression can be rewritten as

$$(\theta^2 - \theta)^3 - t(\theta^2 - \theta) + 1 = 0,$$

which shows that $\beta = \theta^2 - \theta$ is a root of a monic cubic polynomial with coefficients in \mathbf{Q} . Since this polynomial has no integer root, it is irreducible over \mathbb{Z} by the rational root test. Therefore,

$$\mathbf{Q}(\beta) = \mathbf{Q}(\theta^2 - \theta)$$

is a cubic subfield of $L_t = \mathbf{Q}(\theta)$.

Proof of (iii). Computation of the discriminant. Let

$$h(x; t) = x^6 - 3x^5 + 3x^4 - x^3 - tx^2 + tx + 1.$$

We first compute the derivative:

$$h'(x) = 6x^5 - 15x^4 + 12x^3 - 3x^2 - 2tx + t.$$

Since $\deg h = 6$, the discriminant is

$$\Delta(h) = (-1)^{\frac{6-5}{2}} \text{Res}(h, h') = (-1)^{15} \text{Res}(h, h') = -\text{Res}(h, h').$$

The resultant $\text{Res}(h, h')$ is the determinant of the 11×11 Sylvester matrix $S(h, h')$, constructed from the coefficients of h and h' .

Thus,

$$\text{Res}(h, h') = \det S(h, h').$$

Evaluating this determinant and simplifying in $\mathbb{Z}[t]$ gives

$$\text{Res}(h, h') = 256t^7 + 1008t^6 - 3456t^4 - 13608t^3 + 11664t + 45927.$$

Since $\Delta(h) = -\text{Res}(h, h')$, we obtain

$$\Delta(h) = -256t^7 - 1008t^6 + 3456t^4 + 13608t^3 - 11664t - 45927$$

This expression factorizes as

$$\Delta(h) = -(16t + 63)(4t^3 - 27)^2.$$

The proof of monogeneity follows a similar approach to that used in the previous theorem, utilizing the properties of the discriminant and the structure of the polynomial to conclude that the number field defined by this polynomial is monogenic. □

Remark 3.3. *In our context, applying Siegel's theorem [9] implies that for any fixed integer t , there exist only finitely many integers n such that the fields generated by the sextic polynomials discussed here are isomorphic. Consequently, both families of sextic fields considered in this paper give rise to infinitely many **non-isomorphic** number fields having the described properties.*

Conclusion

In this paper, we studied two families of irreducible sextic polynomials and showed that they give rise to infinite families of exceptional and monogenic number fields. One family contains real quadratic subfields of the form $\mathbf{Q}(\sqrt{t^2 - 4})$, while the other contains cubic subfields generated by expressions like $\theta^2 - \theta$.

These examples demonstrate that exceptional and monogenic fields are more common in well-structured families than previously thought. In future work, it would be interesting to explore the Galois groups of these fields, extend the methods to polynomials of higher degree, or study similar phenomena over other base fields.

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