

An Overview of Monomorphism Categories

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ABSTRACT

In this survey, the author collects and presents several of his recent results and joint works on monomorphism categories, covering various topics in representation theory. We try to show that how one can approach the monomorphism categories via functor categories. In the representation-finite case, we are essentially dealing with module categories, so well-understood knowledge from module categories can be transferred to monomorphism categories. In particular, this approach is effective for developing a covering theory for monomorphism categories. The results in Section 3, which provide various equivalences between the monomorphism category and the category of finitely presented covariant functors, are taken from an unpublished paper by the author.

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1 Introduction

Monomorphism categories were first introduced by G. Birkhoff in 1934. Recently, their importance has been investigated from various points of view, since they provide a framework to address open problems in linear algebra by means of methods and results from homological algebra, combinatorics, and geometry. As an important application in data

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science, note that the systems of complexes that give rise to persistence homology forms a monomorphism category.

Throughout, let Λ denote an Artin algebra. The morphism category $\mathbf{H}(\Lambda)$ has as objects the morphisms f in $\text{mod-}\Lambda$, and morphisms are given by commutative diagrams. The category $\mathbf{H}(\Lambda)$ is abelian; indeed, it is equivalent to the category of finitely generated modules over the triangular matrix algebra

$$T_2(\Lambda) = \begin{pmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{pmatrix}.$$

Let $\mathcal{S}(\Lambda)$ be the full subcategory of $\mathbf{H}(\Lambda)$ consisting of all monomorphisms. This category is called the *monomorphism category* or *submodule category*. It inherits an exact structure from $\mathbf{H}(\Lambda)$ since it is closed under extensions. The study of submodule categories goes back to Birkhoff, who studied $\mathcal{S}(\Lambda)$ for $\Lambda = \mathbb{Z}/(p^n)$, where p is a prime number and \mathbb{Z} denotes the ring of integers. Later, Simson studied the case $\Lambda = k[x]/(x^n)$ and explored the tame–wild dichotomy for the monomorphism category $\mathcal{S}(k[x]/(x^n))$ in terms of the parameter n .

More generally, in place of the abelian category $\text{mod-}\Lambda$, one can define the monomorphism category $\mathcal{S}(\Xi)$ associated with an abelian category Ξ . Such a general form of monomorphism categories leads to interesting applications in algebraic geometry. For instance, when Ξ is the category of graded modules over the graded algebra $k[x]/(x^n)$ with $\deg(x) = 1$, the corresponding monomorphism category is denoted by $\mathcal{S}(\tilde{n})$. It is a Frobenius exact category, so its stable category, denoted by $\overline{\mathcal{S}(\tilde{n})}$, is triangulated. An important result due to Kussin, Lenzing, and Meltzer states that $\overline{\mathcal{S}(\tilde{n})}$ is triangle equivalent to the stable category of vector bundles on the weighted projective line of type $(2, 3, p)$.

Recently, Ringel and Schmidmeier [36] investigated the representation type of $\mathcal{S}(\Lambda)$ and developed an Auslander–Reiten theory for submodule categories. They provided an explicit method to compute the (relative) Auslander–Reiten translation in $\mathcal{S}(\Lambda)$. Among their results, they showed that the sixth power of the Auslander–Reiten translation in $\mathcal{S}(\Lambda)$ is invariant whenever Λ is a commutative uniserial algebra. Ringel and Zhang [35] established a remarkable connection between the monomorphism categories of factor algebras of $k[x]$ by a power of x and the preprojective algebras of type A . The latter were introduced by Gelfand and Ponomarev and are now of great interest—for example, they form an important tool in the study of quantum groups (Lusztig) and cluster algebras (Geiss–Leclerc–Schröer). In my recent work [15], this connection was generalized to a wide range of cases between certain kinds of monomorphism categories and relative stable Auslander algebras, which marked the beginning of my research on monomorphism categories. The theory of Ringel and Schmidmeier has also been generalized by Bao-Lin Xiong, Pu Zhang, and Yue-Hui Zhang [37] to the category of $(n - 1)$ -sequences

$$A_1 \xrightarrow{\phi_1} A_2 \longrightarrow \cdots \longrightarrow A_n,$$

where all ϕ_i are monomorphisms.

In this survey, we will collect the main results of several of the author’s joint works on monomorphism categories. To study monomorphism categories, we have tried to approach

them via functor categories, which are abelian. In the representation-finite case, we are essentially dealing with module categories, so well-understood knowledge from module categories can be transferred to monomorphism categories. In particular, this approach is effective for developing a covering theory for monomorphism categories (see Section 8 for more details).

Recently, we have also become interested in other subcategories of the morphism category, not only monomorphism categories. In [19], we studied the Auslander–Reiten theory of the entire morphism category. This perspective allows one to reinterpret some notions from the module category in terms of Auslander–Reiten theory in the morphism category. For example, in [20, Section 5], one sees how the duality functor can be determined by the Auslander–Reiten translation in the morphism category.

In another project (see [24, 23]), we began studying the morphism category of projective modules, another subcategory $\mathcal{P}(\Lambda)$ of the morphism category. We discovered that τ -tilting theory in $\text{mod-}\Lambda$ is initiated from tilting theory in $\mathcal{P}(\Lambda)$. We showed that there exists an injection from Morita equivalence classes of Artin algebras to equivalence classes of 0-Auslander exact categories. Under this injection, one can analyze $\mathcal{P}(\Lambda)$, which has favorable homological properties, and then transfer this information back to the study of $\text{mod-}\Lambda$. A similar observation was made by Auslander, who used the category $((\text{mod-}\Lambda)^{\text{op}}, \mathcal{A}b)$ of contravariantly finitely presented functors from $\text{mod-}\Lambda$ to the category of abelian groups $\mathcal{A}b$ (which has global dimension 2) to discover certain properties of $\text{mod-}\Lambda$. By contrast, $\mathcal{P}(\Lambda)$ is an exact category of global dimension 1, and hence from a homological perspective it is “one dimension smaller”.

This survey is organized so that each section, except Section 3, carries the same title as the author’s papers on monomorphism categories and presents selected results from those works. For proofs and additional background, we refer to the original papers. We highlight that the results in the last section are outlined from ongoing projects or from works that have been submitted for publication. The results in Section 3 are taken from the unpublished paper [14].

2 From subcategories to the entire module categories

In [15], we introduce two functors and study their behavior with respect to Auslander–Reiten theory. As an important application, we show how one can estimate the Auslander–Reiten theory of certain subcategories—particularly the subcategory of Gorenstein projective objects, which is our main interest—by using the Auslander–Reiten theory of abelian categories, which is often easier to analyze.

2.1 The first functor

This functor is inspired by one of the two functors considered in [35]. In [15], we formulate it in a more general form and in a functorial framework. Throughout this section, let \mathcal{A} be an abelian category with enough projectives. Following [34], a subcategory \mathcal{C} of \mathcal{A} is

called *quasi-resolving* if

- it contains the projective objects of \mathcal{A} ,
- it is closed under finite direct sums, and
- it is closed under kernels of epimorphisms in \mathcal{C} .

Moreover, a quasi-resolving subcategory \mathcal{C} of \mathcal{A} is called *resolving* if it is also closed under direct summands and extensions.

For a given subcategory \mathcal{C} of an abelian category \mathcal{A} , we associate the subcategory $\mathcal{S}_{\mathcal{C}}(\mathcal{A})$ of the morphism category $\mathbf{H}(\mathcal{A})$ consisting of all morphisms $(A \xrightarrow{f} B)$ such that:

- (i) f is a monomorphism,
- (ii) A , B , and $\text{Cok } f$ belong to \mathcal{C} .

In the case $\mathcal{A} = \text{mod-}\Lambda$, for $\mathcal{X} \subseteq \text{mod-}\Lambda$ we denote $\mathcal{S}_{\mathcal{X}}(\text{mod-}\Lambda)$ by $\mathcal{S}_{\mathcal{X}}(\Lambda)$, and simply $\mathcal{S}(\Lambda)$ when $\mathcal{X} = \text{mod-}\Lambda$. The latter one is called the monomorphism (submodule) category over Λ

We now define a functor

$$\Psi_{\mathcal{C}} : \mathcal{S}_{\mathcal{C}}(\mathcal{A}) \longrightarrow \text{mod-}\underline{\mathcal{C}}$$

with respect to a subcategory \mathcal{C} of \mathcal{A} as follows.

Construction 2.1. *Let $(A \xrightarrow{f} B)$ be an object of $\mathcal{S}_{\mathcal{C}}(\mathcal{A})$. Then we have a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow \text{Cok } f \longrightarrow 0$$

in \mathcal{A} . This induces the following short exact sequence in $\text{mod-}\underline{\mathcal{C}}$:

$$(*) \quad 0 \longrightarrow (-, A) \xrightarrow{(-, f)} (-, B) \longrightarrow (-, \text{Cok } f) \longrightarrow F \longrightarrow 0.$$

In fact, $(*)$ corresponds to a projective resolution of F in $\text{mod-}\underline{\mathcal{C}}$. We define $\Psi_{\mathcal{C}}(A \xrightarrow{f} B) := F$.

For morphisms: let $\sigma = (\sigma_1, \sigma_2)$ be a morphism from $(A \xrightarrow{f} B)$ to $(A' \xrightarrow{f'} B')$. It yields the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & \text{Cok } f \longrightarrow 0 \\ & & \downarrow \sigma_1 & & \downarrow \sigma_2 & & \downarrow \sigma_3 \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \longrightarrow & \text{Cok } f' \longrightarrow 0 \end{array}$$

in \mathcal{A} . By applying the Yoneda functor, we obtain a commutative diagram in $\text{mod-}\underline{\mathcal{C}}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (-, A) & \xrightarrow{(-, f)} & (-, B) & \longrightarrow & (-, \text{Cok } f) \longrightarrow F \longrightarrow 0 \\ & & \downarrow (-, \sigma_1) & & \downarrow (-, \sigma_2) & & \downarrow (-, \sigma_3) \\ 0 & \longrightarrow & (-, A') & \xrightarrow{(-, f')} & (-, B') & \longrightarrow & (-, \text{Cok } f') \longrightarrow F' \longrightarrow 0 \end{array}$$

We define $\Psi_{\mathcal{C}}(\sigma) := \overline{(-, \sigma_3)}$.

Theorem 2.2. *Let \mathcal{A} be an abelian category with enough projectives, and let \mathcal{C} be a quasi-resolving subcategory of \mathcal{A} . Consider the full subcategory \mathcal{V} of $\mathcal{S}_{\mathcal{C}}(\mathcal{A})$ formed by all finite direct sums of objects of the form $(C \xrightarrow{1} C)$ and $(0 \rightarrow C)$, where C runs through the objects of \mathcal{C} . Then the functor $\Psi_{\mathcal{C}}$, defined in Construction 2.1, induces an equivalence of categories*

$$\mathcal{S}_{\mathcal{C}}(\mathcal{A})/\mathcal{V} \simeq \text{mod-}\underline{\mathcal{C}}.$$

In particular, if $\mathcal{C} \subseteq \text{mod-}\Lambda$ is of finite representation type, then the following conditions are equivalent:

- (1) The relative stable Auslander algebra $\text{Aus}(\underline{\mathcal{C}})$ is of finite representation type, i.e., $\text{Aus}(\underline{\mathcal{C}}) = \text{End}_{\underline{\mathcal{C}}}(\underline{M})$ for some additive generator M of \mathcal{C} .
- (2) The subcategory $\mathcal{S}_{\mathcal{C}}(\Lambda)$ of $\text{mod-}T_2(\Lambda)$ is of finite representation type.

By applying this to $\mathcal{C} = \text{Gprj-}\Lambda$, we obtain

$$\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda) \simeq \text{Gprj-}T_2(\Lambda).$$

Thus, for a CM-finite algebra Λ , the algebra $T_2(\Lambda)$ is CM-finite if and only if the stable Cohen–Macaulay Auslander algebra of Λ is of finite representation type. Moreover, if Λ is self-injective of finite representation type, then $T_2(\Lambda)$ is CM-finite if and only if the stable Auslander algebra of Λ is representation-finite.

We now give the following application of this observation.

Example 2.3. *Let $A = k\mathcal{Q}/I$ be a quadratic monomial algebra, i.e., the ideal I is generated by paths of length two. By [6, Theorem 5.7], we have*

$$\underline{\text{Gprj-}}A \simeq \mathcal{T}_1 \times \cdots \times \mathcal{T}_n,$$

where each \mathcal{T}_i is a triangulated category whose underlying category is equivalent to a semisimple abelian category $\text{mod-}k^{d_i}$ for some natural number d_i . Hence, $\text{mod-}\underline{\text{Gprj-}}A$ is a semisimple abelian category. This implies that $\text{Aus}(\underline{\text{Gprj-}}A)$ is semisimple Artinian, and therefore the algebra $T_2(A)$ is CM-finite.

2.2 Exchange between the almost split sequences in $\mathcal{S}_{\mathcal{X}}(\Lambda)$ and $\text{mod-}\underline{\mathcal{X}}$

Setup 2.4. *Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be contravariantly finite and resolving. Assume further that \mathcal{X} has enough injectives and that $\mathcal{S}_{\mathcal{X}}(\Lambda)$ admits almost split sequences.*

In the following construction, we explain how the almost split sequences in $\text{mod-}\underline{\mathcal{X}}$ can be computed from those in $\mathcal{S}_{\mathcal{X}}(\Lambda)$.

Let \mathcal{X} be as in Setup 2.4, and let H be a non-projective indecomposable functor in $\text{mod-}\underline{\mathcal{X}}$. Then s_H is also non-projective indecomposable in $\mathcal{S}_{\mathcal{X}}(\Lambda)$. Hence, by assumption, there exists an almost split sequence in $\mathcal{S}_{\mathcal{X}}(\Lambda)$ ending at s_H , namely,

$$\epsilon : 0 \longrightarrow \begin{array}{c} X_1 \\ \downarrow d \\ X_2 \end{array} \xrightarrow[\phi_2]{\phi_1} \begin{array}{c} Z_1 \\ \downarrow \iota \\ Z_2 \end{array} \xrightarrow[\psi_2]{\psi_1} \begin{array}{c} A_H \\ \downarrow s_H \\ B_H \end{array} \longrightarrow 0$$

This short exact sequence induces the following commutative diagram in $\text{mod-}\underline{\mathcal{X}}$:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & 0 & (*) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & (-, X_1) & \xrightarrow{(-, d)} & (-, X_2) & \longrightarrow & (-, \text{Cok } d) & \longrightarrow & F & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow f & \\ 0 & \longrightarrow & (-, X_1 \oplus A_H) & \xrightarrow{(-, h)} & (-, X_2 \oplus B_H) & \longrightarrow & (-, \text{Cok } d \oplus C_H) & \longrightarrow & G & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow g & \\ 0 & \longrightarrow & (-, A_H) & \xrightarrow{(-, s_H)} & (-, B_H) & \longrightarrow & (-, C_H) & \longrightarrow & H & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & & 0 & & 0 & & 0 & & 0 & \end{array}$$

According to Construction 2.1, the sequence in the rightmost column (*) is the image of ϵ under the functor Ψ_C . We denote it by S_H . It was first shown that this is indeed a short exact sequence, and hence it can be regarded as a short exact sequence in $\text{mod-}\underline{\mathcal{X}}$. Moreover, in [15, Proposition 5.6] it was proved that S_H is an almost split sequence in $\text{mod-}\underline{\mathcal{X}}$.

As a direct consequence of the exchange of almost split sequences described above, we obtain the following result:

Proposition 2.5. *Let \mathcal{X} be as in Setup 2.4. Then $\text{mod-}\underline{\mathcal{X}}$ has almost split sequences.*

Conversely, one can also start from an almost split sequence in $\text{mod-}\underline{\mathcal{X}}$ and construct a corresponding almost split sequence in $\mathcal{S}_{\mathcal{X}}(\Lambda)$.

2.3 Gorenstein projective functors

Here, we have a characterization of non-projective Gorenstein projective indecomposable objects in $\text{mod-}\underline{\mathcal{X}}$ via their minimal projective resolutions in $\text{mod-}\underline{\mathcal{X}}$. We will observe how the monomorphism category $\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)$ are related to the study the Gorenstein projective to other abelian categories not only $\text{mod-}T_2(\Lambda)$.

Theorem 2.6. [15, Theorem 4.5] *Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be closed under projective cosyzygies. Assume that there is $n \geq 0$ such that $\Omega^n(\mathcal{X})$ is contained in $\text{Gprj-}\Lambda$. Let F be*

a non-projective indecomposable functor in $\text{mod-}\underline{\mathcal{X}}$ with the following minimal projective resolution

$$0 \rightarrow (-, A) \rightarrow (-, B) \rightarrow (-, C) \rightarrow F \rightarrow 0.$$

Then F is a Gorenstein projective object in $\text{mod-}\underline{\mathcal{X}}$ if and only if A, B and C are Gorenstein projective modules in $\text{mod-}\Lambda$.

The above characterization might not be true for all the Gorenstein projective functors. In fact, consider an indecomposable module $X \in \mathcal{X}$ which does not belong to $\text{Gprj-}\Lambda$, we have the following minimal projective resolution of $(-, \underline{X}), 0 \rightarrow (-, \Omega_\Lambda(X)) \rightarrow (-, P_X) \rightarrow (-, X) \rightarrow (-, \underline{X}) \rightarrow 0$ in $\text{mod-}\mathcal{X}$. But X does not lie in $\text{Gprj-}\Lambda$.

We point out that for a quasi-resolving subcategory \mathcal{X} the condition “ $\Omega^n(\mathcal{X})$ is contained in $\text{Gprj-}\Lambda$ and closed under cosyzygies” is called (\mathbf{G}_n) in [34].

Here, we provide a characterization of non-projective, Gorenstein projective indecomposable objects in $\text{mod-}\underline{\mathcal{X}}$ in terms of their minimal projective resolutions in $\text{mod-}\mathcal{X}$. We also observe how the monomorphism category $\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)$ is related to the study of Gorenstein projective objects in other abelian categories, not only in $\text{mod-}T_2(\Lambda)$.

2.4 The second functor

Let \mathcal{C}' be a full subcategory of an additive category \mathcal{C} . As discussed in [3, Section 3], the restriction functor $\text{res} : (\text{mod } \mathcal{C}) \rightarrow (\text{mod } \mathcal{C}')$ admits left and right adjoints. Here $(\text{mod } \mathcal{C})$ and $(\text{mod } \mathcal{C}')$ denote the category of all contravariant functors from \mathcal{C} and \mathcal{C}' to the category of abelian groups, respectively. We denote the left adjoint functor of the restriction functor by ${}_{\mathcal{C}'}\mathcal{Y}_{\mathcal{C}}$. Here, we explain the left adjoint functor in more details as follows: let F be a finitely presented functor over \mathcal{C}' . So there is an exact sequence

$$(-, C_2) \rightarrow (-, C_1) \rightarrow F \rightarrow 0$$

with C_i in \mathcal{C}' , $i = 1, 2$. We can naturally construct the functor \tilde{F} in $\text{mod-}\mathcal{C}$, by defining $\tilde{F}(C) := \text{Cok}(\text{Hom}_{\mathcal{C}}(C, C_2) \rightarrow \text{Hom}_{\mathcal{C}}(C, C_1))$ for any C in \mathcal{C} . Define ${}_{\mathcal{C}'}\mathcal{Y}_{\mathcal{C}}(F) = \tilde{F}$.

Let $\sigma : F \rightarrow G$ be a morphism in $\text{mod-}\mathcal{C}'$. The morphism σ can be lifted as the following to their existing projective presentations

$$\begin{array}{ccccccc} (-, C_2) & \longrightarrow & (-, C_1) & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow (-, f_2) & & \downarrow (-, f_1) & & \downarrow \sigma & & \\ (-, C'_2) & \longrightarrow & (-, C'_1) & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

in $\text{mod-}\mathcal{C}'$. For each $C \in \mathcal{C}$, $\tilde{\sigma}_C : \tilde{F}(C) \rightarrow \tilde{G}(C)$ is defined as the following

$$\begin{array}{ccccccc} \text{Hom}_{\mathcal{C}}(C, C_2) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, C_1) & \longrightarrow & \tilde{F}(C) & \longrightarrow & 0 \\ \downarrow (C, f_2) & & \downarrow (C, f_1) & & \downarrow \tilde{\sigma}_C & & \\ \text{Hom}_{\mathcal{C}}(C, C'_2) & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, C'_1) & \longrightarrow & \tilde{G}(C) & \longrightarrow & 0. \end{array}$$

Define ${}_{\mathcal{C}'}\mathcal{Y}_{\mathcal{C}}(\sigma) = \tilde{\sigma}$. By [3], the functor ${}_{\mathcal{C}'}\mathcal{Y}_{\mathcal{C}}$ is fully faithful.

Setup 2.7. Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be a contravariantly finite and quasi-resolving subcategory being closed under projective cosyzygies. Also assume that there is $n \geq 0$ such that $\Omega^n(\mathcal{X})$ is contained in $\text{Gprj-}\Lambda$. Denote $\mathcal{Y} = \mathcal{X} \cap \text{Gprj-}\Lambda$. It is clear that \mathcal{Y} is again a quasi-resolving subcategory of $\text{mod-}\Lambda$. Note that by [34, Theorem 1.4], the equality $\Omega^n(\mathcal{X}) = \mathcal{X} \cap \text{Gprj-}\Lambda$ holds.

Assume \mathcal{X} and \mathcal{Y} are the same as in Setup 2.7. Specializing in the above Construction for $\underline{\mathcal{Y}} \subseteq \underline{\mathcal{X}}$, we reach the extension functor $\underline{\mathcal{Y}}\mathcal{I}_{\underline{\mathcal{X}}} : \text{mod-}\underline{\mathcal{Y}} \rightarrow \text{mod-}\underline{\mathcal{X}}$. Let us remark that by identifying $\text{mod-}\underline{\mathcal{Y}}$ and $\text{mod-}\underline{\mathcal{X}}$, respectively, as subcategories of $\text{mod-}\mathcal{Y}$ and $\text{mod-}\mathcal{X}$, we can consider $\underline{\mathcal{Y}}\mathcal{I}_{\underline{\mathcal{X}}}$ as a restriction of the extension functor $\mathcal{Y}\mathcal{I}_{\mathcal{X}} : \text{mod-}\mathcal{Y} \rightarrow \text{mod-}\mathcal{X}$.

Proposition 2.8. Let \mathcal{X} and \mathcal{Y} be as in Setup 2.7. Then the extension functor $\underline{\mathcal{Y}}\mathcal{I}_{\underline{\mathcal{X}}}$, defined in the above, is an exact functor. Moreover, its essential image is contained in $\text{Gprj-}\underline{\mathcal{X}}$.

We know by the above proposition the essential image of $\underline{\mathcal{Y}}\mathcal{I}_{\underline{\mathcal{X}}}$ is contained in $\text{Gprj-}\underline{\mathcal{X}}$. Denote by $\underline{\mathcal{Y}}\hat{\mathcal{I}}_{\underline{\mathcal{X}}} : \text{mod-}\underline{\mathcal{Y}} \rightarrow \text{Gprj-}\underline{\mathcal{X}}$ the induced functor.

Definition 2.9. A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is called almost dense if all but finitely many indecomposable objects, up to isomorphism, are in the essential image of F .

Proposition 2.10. Let \mathcal{X} and \mathcal{Y} be as in Setup 2.7. Then the following assertions hold.

- (i) The induced functor $\underline{\mathcal{Y}}\hat{\mathcal{I}}_{\underline{\mathcal{X}}} : \text{mod-}\underline{\mathcal{Y}} \rightarrow \text{Gprj-}\underline{\mathcal{X}}$ is an exact functor which is fully faithful. Here assume that $\text{Gprj-}\underline{\mathcal{X}}$ gets a natural exact structure from $\text{mod-}\underline{\mathcal{X}}$.
- (ii) The essential image of $\underline{\mathcal{Y}}\hat{\mathcal{I}}_{\underline{\mathcal{X}}}$ contains all indecomposable functors in $\text{Gprj-}\underline{\mathcal{X}}$ except the projective indecomposable $(-, \underline{X})$, where X is an indecomposable module in \mathcal{X} but not in \mathcal{Y} .
- (iii) If \mathcal{X} is of finite representation type, then $\underline{\mathcal{Y}}\hat{\mathcal{I}}_{\underline{\mathcal{X}}}$ is almost dense.

Specializing the above to the case when \mathcal{X} is the subcategory of Gorenstein projective modules over a Gorenstein Artin algebra, we get the following corollaries. For simplicity, we denote the functor $\underline{\text{Gprj-}}\hat{\mathcal{I}}_{\underline{\text{mod-}}\Lambda}$ by $\underline{\text{Gprj-}}\hat{\mathcal{I}}_{\Lambda}$. Then, we have the following corollaries:

Corollary 2.11. Let Λ be an n -Gorenstein algebra. Then

- (1) The functor $\underline{\text{Gprj-}}\hat{\mathcal{I}}_{\Lambda} : \text{mod-}\underline{\text{Gprj-}}\Lambda \rightarrow \text{Gprj-}\underline{\text{mod-}}\Lambda$ is an exact functor which is fully faithful.
- (2) The essential image of $\underline{\text{Gprj-}}\hat{\mathcal{I}}_{\Lambda}$ contains all indecomposable functors in $\text{Gprj-}\underline{\text{mod-}}\Lambda$ but indecomposable functors $(-, \underline{X})$ such that X is not isomorphic to a Gorenstein projective indecomposable module.
- (3) If Λ is of finite representation type, then the functor $\underline{\text{Gprj-}}\hat{\mathcal{I}}_{\Lambda} : \text{mod-}\underline{\text{Gprj-}}\Lambda \rightarrow \text{Gprj-}\underline{\text{mod-}}\Lambda$ is almost dense.

Corollary 2.12. *Let Λ be a Gorenstein algebra of finite representation type. Then the following conditions are equivalent.*

- (1) $T_2(\Lambda)$ is CM-finite and Gorenstein;
- (2) The stable Cohen-Macaulay Auslander algebra of Λ is a representation-finite self-injective algebra;
- (3) The stable Auslander algebra of Λ is CM-finite and Gorenstein.

2.5 Exchange between the almost split sequences in $\text{mod-}\underline{\mathcal{X}} \cap \text{Gprj-}\underline{\Lambda}$ and $\text{Gprj-}\underline{\mathcal{X}}$

Recall from the previous section that there is the functor $\underline{\mathcal{Y}}\hat{\mathcal{Y}}_{\underline{\mathcal{X}}} : \text{mod-}\underline{\mathcal{Y}} \rightarrow \text{Gprj-}\underline{\mathcal{X}}$ which is exact and fully faithful. For simplicity, set $\tilde{F} := \underline{\mathcal{Y}}\hat{\mathcal{Y}}_{\underline{\mathcal{X}}}(F)$ for any F in $\text{mod-}\underline{\mathcal{Y}}$ and also $\tilde{f} := \underline{\mathcal{Y}}\hat{\mathcal{Y}}_{\underline{\mathcal{X}}}(f)$ for any morphism f in $\text{mod-}\underline{\mathcal{Y}}$. In the next results, we shall show the functor $\underline{\mathcal{Y}}\hat{\mathcal{Y}}_{\underline{\mathcal{X}}}$ preserves the almost split sequences.

Proposition 2.13. *Let \mathcal{X} and \mathcal{Y} be the same as in Set up 2.7. Assume $\eta : 0 \rightarrow F \xrightarrow{f} G \xrightarrow{g} H \rightarrow 0$ is a short exact sequence in $\text{mod-}\underline{\mathcal{Y}}$. Then η is an almost split sequence in $\text{mod-}\underline{\mathcal{Y}}$ if and only if its image under $\underline{\mathcal{Y}}\hat{\mathcal{Y}}_{\underline{\mathcal{X}}}$*

$$\tilde{\eta} : 0 \rightarrow \tilde{F} \xrightarrow{\tilde{f}} \tilde{G} \xrightarrow{\tilde{g}} \tilde{H} \rightarrow 0$$

so is in $\text{Gprj-}\underline{\mathcal{X}}$.

As an application of our result in conjunction with [36, Corollary 6.5] we have:

Corollary 2.14. *Let Λ be a commutative Nakayama algebra. Let Γ denote the stable Auslander algebra of Λ , i.e. $\Gamma := \underline{\text{End}}(M)$, where M is a representation generator of $\text{mod-}\Lambda$. Then $\tau_{\Gamma}^{\text{st}}(N) \simeq N$ for each non-projective indecomposable N in $\text{mod-}\Gamma$.*

2.6 Auslander-Reiten quiver

Let \mathcal{C} be a full extension-closed subcategory of an abelian Krull–Schmidt category \mathcal{A} . Auslander and Reiten defined for \mathcal{C} an associated valued quiver

$$\Gamma_{\mathcal{C}} = (\Gamma_{\mathcal{C}}^0, \Gamma_{\mathcal{C}}^1),$$

called the *Auslander–Reiten quiver* of \mathcal{C} , as follows:

The vertices $\Gamma_{\mathcal{C}}^0$ are in one-to-one correspondence with the isomorphism classes of objects in \mathcal{C} , usually denoted by $[M]$ (or simply M) for an indecomposable object $M \in \mathcal{C}$.

There is an arrow $[M] \rightarrow [N]$ in $\Gamma_{\mathcal{C}}^1$ with valuation (a, b) if there exists a minimal right almost split morphism

$$M^a \oplus X \longrightarrow N$$

such that X has no direct summand isomorphic to M in \mathcal{C} , and a minimal left almost split morphism

$$M \longrightarrow N^b \oplus Y$$

in \mathcal{C} .

We denote by $\Gamma_{\mathcal{C}}^s$ the valued quiver, called the *stable Auslander–Reiten quiver* of \mathcal{C} , obtained by removing all vertices corresponding to Ext-projective and Ext-injective indecomposable objects in \mathcal{C} together with all arrows attached to them.

In order to simplify notation, when $\mathcal{C} = \text{mod-}\mathcal{D}$ we write $\Gamma_{\mathcal{D}}$ instead of $\Gamma_{\text{mod-}\mathcal{D}}$.

The two functors mentioned earlier preserve almost split sequences. This property is reflected in the corresponding Auslander–Reiten quivers. In the next result, we observe how one can compare the Auslander–Reiten quiver of the subcategory of Gorenstein projective modules with the Auslander–Reiten quiver of an abelian category.

Theorem 2.15. [15, Theorem 6.2] *Let Λ be a Gorenstein algebra. The Auslander–Reiten quiver $\Gamma_{\underline{\text{Gprj}}-\Lambda}$ of $\text{mod-}\underline{\text{Gprj}}-\Lambda$ embeds into both $\Gamma_{\text{Gprj-}T_2(\Lambda)}$ and $\Gamma_{\text{Gprj-}\underline{\text{mod}}-\Lambda}$ as a full valued subquiver, i.e.,*

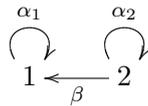
$$\Gamma_{\text{Gprj-}T_2(\Lambda)} \hookleftarrow \Gamma_{\underline{\text{Gprj}}-\Lambda} \hookrightarrow \Gamma_{\text{Gprj-}\underline{\text{mod}}-\Lambda}.$$

Moreover, the stable Auslander–Reiten quivers coincide:

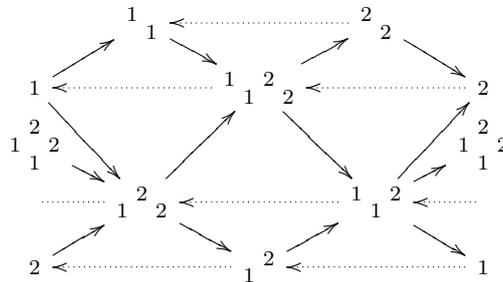
$$\Gamma_{\underline{\text{Gprj}}-\Lambda}^s = \Gamma_{\text{Gprj-}\underline{\text{mod}}-\Lambda}^s.$$

The next two examples illustrate the above theorem

Example 2.16. *Let $\Lambda = T_2(k[x]/(x^2))$, $k[x]$ be the polynomial ring in one variable x with coefficients in k . The algebra Λ is given by the quiver*

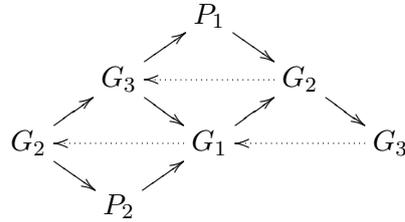


The AR-quiver of Λ is as below:

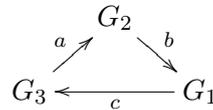


where the vertices are displayed by the composition series. Set $A = \text{Aus}(\underline{\text{mod}}-\Lambda)$ and $B = \text{Aus}(\underline{\text{Gprj}}-\Lambda)$. The indecomposable modules with composition series $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$, $\begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ are Gorenstein projective modules; label those with G_1, G_2 and G_3 respectively. Also, let P_1 and P_2 denote respectively the projective indecomposable modules corresponding to the

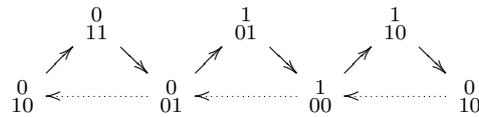
vertices 1 and 2 in the bound quiver of Λ . The Auslander-Reiten quiver $\Gamma_{\text{Gprj-}\Lambda}$ of the subcategory $\text{Gprj-}\Lambda$ is given as the following



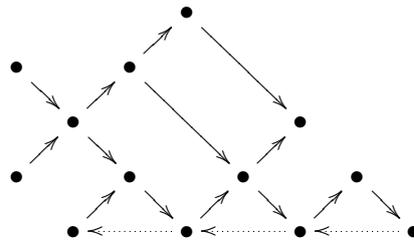
Hence, the stable Cohen-Macaulay Auslander algebra B of Λ is given by the quiver



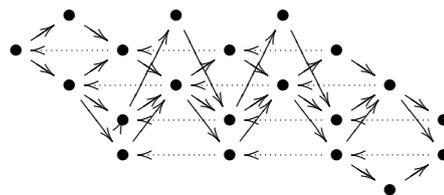
with the relations $ab = bc = ca = 0$. The Auslander-Reiten quiver of B is of the form



According to Theorem 2.15, we can obtain $\Gamma_{\text{Gprj-}\Lambda}$ from Γ_B by adding the missing vertices, as described below:

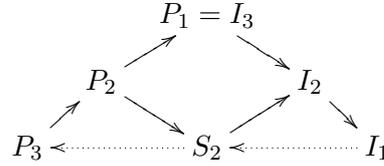


Again by use of Theorem 2.15, we obtain as below the AR-quiver of $\text{Gprj-T}_2(\Lambda)$ by adding the missing vertices to $\Gamma_{\text{Gprj-}\Lambda}$.



Example 2.17. Let Λ be the path algebra of the quiver $A_3 : 1 \rightarrow 2 \rightarrow 3$. The Auslander-

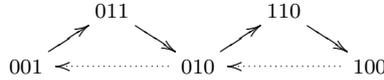
Reiten quiver Γ_Λ of Λ is



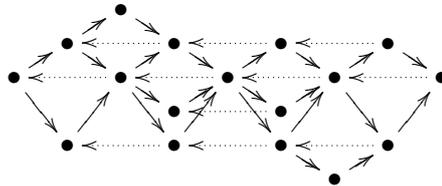
As usual, the P_i, I_i and S_i , respectively, show the indecomposable projective, injective and simple modules corresponding to the vertex i . Then the stable Auslander algebra $A = \text{Aus}(\underline{\text{mod}}-\Lambda)$ is given by the following quiver

$$I_1 \xrightarrow{\alpha} I_2 \xrightarrow{\beta} S_2$$

bound by $\alpha\beta = 0$. The Auslander-Reiten quiver of A is of the form



Here, by applying Theorem 2.15, we can obtain the Auslander-Reiten quiver of $\mathcal{S}(\Lambda)$ as follows. We should mention that, for the left embedding in the theorem, it is not necessary to assume that Λ is a Gorenstein algebra.



3 Some more equivalences

In this section, in the spirit of the equivalence given in Section 2 between the monomorphism category and the functor category, we will provide in the last theorem of this section some further variants of this kind of equivalence. The interesting point of these equivalences is that modules over both sides of a ring are involved. The results in this section are taken from an unpublished paper [14] by the author, which is available on arXiv.

The two categories in the equivalence of Theorem 2.2 can be naturally dualized. Hence we obtain a dual version of the theorem. To state this dual version, we introduce some notation. For a subcategory \mathcal{X} of \mathcal{A} , let $\mathcal{F}_{\mathcal{X}}(\mathcal{A})$ be the subcategory of $\text{H}(\mathcal{A})$ consisting of morphisms $A \xrightarrow{f} B$ satisfying:

- (1) f is an epimorphism;
- (2) A, B , and $\text{Ker}(f)$ lie in \mathcal{X} .

The functor $\Psi' : \mathcal{F}_{\mathcal{X}}(\mathcal{A}) \rightarrow (\overline{\mathcal{X}}\text{-mod})^{\text{op}}$, defined with respect to the subcategory \mathcal{X} , is constructed similarly to Ψ . It sends an object $(A \xrightarrow{f} B)$ to a functor fitting into the exact sequence

$$0 \longrightarrow (B, -) \xrightarrow{(f, -)} (A, -) \longrightarrow (\text{Ker}(f), -) \longrightarrow F \longrightarrow 0,$$

and morphisms are defined in the obvious way.

Theorem 3.1. *Let \mathcal{A} be an abelian category with enough injectives. Let \mathcal{X} be a subcategory of \mathcal{A} containing $\text{Inj-}\mathcal{A}$, which is covariantly finite and closed under cokernels of monomorphisms. Consider the full subcategory \mathcal{V}' consisting of finite direct sums of objects of the forms $(X \xrightarrow{\text{Id}} X)$ or $(X \rightarrow 0)$, where X runs through all objects of \mathcal{X} . Then the functor Ψ' , defined above, induces the following equivalence of categories:*

$$\mathcal{F}_{\mathcal{X}}(\mathcal{A})/\mathcal{V}' \simeq (\overline{\mathcal{X}}\text{-mod})^{\text{op}}.$$

Proof. The proof is the dual of Theorem 2.2, or can be deduced directly from it by viewing \mathcal{A}^{op} as an abelian category with enough projectives and \mathcal{X}^{op} as a subcategory of \mathcal{A}^{op} satisfying the assumptions of the theorem. \square

For a subcategory \mathcal{X} of \mathcal{A} , we have so far attached two subcategories $\mathcal{F}_{\mathcal{X}}(\mathcal{A})$ and $\mathcal{S}_{\mathcal{X}}(\mathcal{A})$ of $\text{H}(\mathcal{A})$. These two subcategories are related by the kernel and cokernel functors:

$$\begin{aligned} \text{Cok} : \mathcal{S}_{\mathcal{X}}(\mathcal{A}) &\longrightarrow \mathcal{F}_{\mathcal{X}}(\mathcal{A}), & (A \xrightarrow{f} B) &\longmapsto (B \xrightarrow{\text{can}} \text{Coker}(f)), \\ \text{Ker} : \mathcal{F}_{\mathcal{X}}(\mathcal{A}) &\longrightarrow \mathcal{S}_{\mathcal{X}}(\mathcal{A}), & (A \xrightarrow{g} B) &\longmapsto (\text{Ker}(g) \xrightarrow{\text{inc}} A). \end{aligned}$$

The kernel and cokernel functors form a pair of inverse equivalences. Moreover, the subcategories \mathcal{V} and \mathcal{V}' (defined respectively in Theorems 2.2 and 3.1) are preserved by Cok and Ker , respectively. Thus, we obtain a pair of inverse equivalences for the corresponding quotient categories. We apply the same notation for the induced pair of inverse equivalences between $\mathcal{F}_{\mathcal{X}}(\mathcal{A})/\mathcal{V}'$ and $\mathcal{S}_{\mathcal{X}}(\mathcal{A})/\mathcal{V}$.

Let \mathcal{X} be a subcategory of an abelian category \mathcal{A} with enough projectives and injectives. Suppose that \mathcal{X} contains all projective and injective objects, is functorially finite, and is closed under kernels of epimorphisms and cokernels of monomorphisms. Let Φ and Φ' be the following compositions:

$$\begin{aligned} \text{mod-}\underline{\mathcal{X}} &\xrightarrow{\Psi^{-1}} \mathcal{S}_{\mathcal{X}}/\mathcal{V} \xrightarrow{\text{Cok}} \mathcal{F}_{\mathcal{X}}/\mathcal{V}' \xrightarrow{\Psi'} \overline{\mathcal{X}}\text{-mod}, \\ \text{mod-}\underline{\mathcal{X}} &\xleftarrow{\Psi} \mathcal{S}_{\mathcal{X}}/\mathcal{V} \xleftarrow{\text{Ker}} \mathcal{F}_{\mathcal{X}}/\mathcal{V}' \xleftarrow{(\Psi')^{-1}} \overline{\mathcal{X}}\text{-mod}, \end{aligned}$$

where Ψ^{-1} and $(\Psi')^{-1}$ denote the quasi-inverses of the functors Ψ and Ψ' , respectively, as described in Theorems 2.2 and 3.1. Then Φ and Φ' form a pair of inverse equivalences with respect to the subcategory \mathcal{X} .

Proposition 3.2. *Let \mathcal{X} be a functorially finite subcategory of an abelian category \mathcal{A} that has enough projectives and injectives. Assume that \mathcal{X} contains all projective and injective objects of \mathcal{A} , and is closed under kernels of epimorphisms and cokernels of monomorphisms. Then $\text{mod-}\underline{\mathcal{X}}$ and $\overline{\mathcal{X}}\text{-mod}$ are both abelian categories with enough injectives.*

Proof. A duality is an exact functor and sends projective objects to injective ones. Since $\text{mod-}\underline{\mathcal{X}}$ and $\overline{\mathcal{X}}\text{-mod}$ always have enough projectives, the dualities Φ and Φ' defined above imply that both categories also have enough injectives. This proves the assertion. \square

Recall that a subcategory \mathcal{X} of \mathcal{A} is resolving if it contains all projectives, and if for every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} we have $A, C \in \mathcal{X}$ implies $B \in \mathcal{X}$, and $B, C \in \mathcal{X}$ implies $A \in \mathcal{X}$. Coresolving subcategories are defined dually.

Example 3.3. (i) *It is clear that whenever \mathcal{X} is the whole abelian category \mathcal{A} , it satisfies all the conditions of Proposition 3.2. A non-trivial example is the subcategory $\mathcal{P}^{<\infty}(\Lambda)$ of finitely generated modules of finite projective dimension (equivalently, finite injective dimension) over a Gorenstein algebra Λ . It is known that over a Gorenstein ring a module has finite projective dimension if and only if it has finite injective dimension. Hence, $\mathcal{P}^{<\infty}(\Lambda)$ contains all injective modules. Using long exact sequences, it is also straightforward to check that $\mathcal{P}^{<\infty}(\Lambda)$ is closed under kernels of epimorphisms and cokernels of monomorphisms.*

It remains to show that $\mathcal{P}^{<\infty}(\Lambda)$ is functorially finite. Over Gorenstein algebras we have the cotorsion pair $(\text{Gprj-}\Lambda, \mathcal{P}^{<\infty}(\Lambda))$ in $\text{mod-}\Lambda$, which is complete; hence, $\mathcal{P}^{<\infty}(\Lambda)$ is covariantly finite in $\text{mod-}\Lambda$, see e.g. [8] for facts on Gorenstein rings. Since $\mathcal{P}^{<\infty}(\Lambda)$ is coresolving and covariantly finite, it is also contravariantly finite by [31]. Thus, $\mathcal{P}^{<\infty}(\Lambda)$ is functorially finite in $\text{mod-}\Lambda$.

Let $H = H(C, D, \Omega)$ be the 1-Gorenstein algebra attached to a symmetrizable generalized Cartan matrix C , an orientation Ω of C , and a symmetrizer D , see [13]. Let $\text{rep}_{\text{l.f.}}(H)$ denote the subcategory of locally free H -modules. By [13, Theorem 1.2], we have $\mathcal{P}^{<\infty}(H) = \text{rep}_{\text{l.f.}}(H)$, so this example satisfies the conditions of Proposition 3.2.

(ii) *Let Λ be a self-injective algebra. Then the stable category $\underline{\text{mod}}\text{-}\Lambda$ is triangulated, with triangles arising from short exact sequences in $\text{mod-}\Lambda$. Let $\pi : \text{mod-}\Lambda \rightarrow \underline{\text{mod}}\text{-}\Lambda$ be the canonical functor, and let \mathcal{Y} be a triangulated subcategory of $\underline{\text{mod}}\text{-}\Lambda$. Then $\pi^{-1}(\mathcal{Y})$ is a subcategory of $\text{mod-}\Lambda$ that contains all projective-injective modules. Moreover, it is closed under kernels of epimorphisms and cokernels of monomorphisms; this follows from the structure of triangles in $\underline{\text{mod}}\text{-}\Lambda$.*

If, in addition, Λ is of finite representation type, then $\pi^{-1}(\mathcal{Y})$ becomes functorially finite in $\text{mod-}\Lambda$, and hence satisfies the conditions of Proposition 3.2.

(iii) *Assume that $({}^\perp\xi, \xi, \xi^\perp)$ is a cotorsion triple in an abelian category \mathcal{A} with enough projectives and injectives; that is, $({}^\perp\xi, \xi)$ and (ξ, ξ^\perp) are cotorsion pairs in \mathcal{A} . Then, by definition, ξ satisfies the required conditions.*

We describe a way to produce new cotorsion triples from a given one. Let \mathcal{Q} be an acyclic quiver with finitely many vertices, and let $\text{Rep}(\mathcal{Q}, \mathbb{R})$ be the category of

representations of \mathcal{Q} by R -modules and R -homomorphisms. Let $\mathcal{Q}(\xi)$ denote the class of all representations in $\text{Rep}(\mathcal{Q}, R)$ whose vertex modules lie in ξ . By [9, Theorem A], $\mathcal{Q}(\xi)$ is the middle term of a cotorsion triple in $\text{Rep}(\mathcal{Q}, R)$. Although [9, Theorem A] is stated for module categories over rings, the proof works for any abelian category with enough projectives and injectives.

For example, $((\text{mod } R), \text{Inj-}R = \text{Prj-}R, (\text{mod } R))$ is a cotorsion triple whenever R is quasi-Frobenius, so the trivial cotorsion triple yields a new subcategory $\mathcal{Q}(\text{Inj-}R)$ in $\text{Rep}(\mathcal{Q}, R)$ satisfying our conditions. Note that Example (i) is a special case of this construction since $(\text{Gprj-}\Lambda, \mathcal{P}^{<\infty}(\Lambda), \text{Ginj-}\Lambda)$ is a cotorsion triple in $\text{mod-}\Lambda$ when Λ is Gorenstein.

Lemma 3.4. *Let \mathcal{X} be a subcategory of an abelian category \mathcal{A} with enough projective and injective objects, satisfying the assumptions of Proposition 3.2. Then \mathcal{X} is closed under extensions; that is, if*

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

is exact with $X, Z \in \mathcal{X}$, then $Y \in \mathcal{X}$.

Proof. Consider a short exact sequence

$$0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$$

with $X, Y \in \mathcal{X}$. Since \mathcal{A} has enough injectives, there exists a monomorphism $X \rightarrow I$ for some injective object I in \mathcal{A} . Form the pushout to obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & I & \longrightarrow & U & \longrightarrow & Y \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \Omega^{-1}(X) & \xlongequal{\quad} & \Omega^{-1}(X) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since subcategories in this paper are assumed to be closed under finite direct sums, we have

$$U \simeq I \oplus Y \in \mathcal{X},$$

because I and Y both lie in \mathcal{X} . Because \mathcal{X} is closed under kernels of epimorphisms, the short exact sequence

$$0 \rightarrow Z \rightarrow U \rightarrow Y \rightarrow 0$$

implies that $Z \in \mathcal{X}$. This completes the proof. □

As an immediate application of the above lemma, we obtain:

Corollary 3.5. *Let \mathcal{X} be a subcategory of an abelian category \mathcal{A} with enough projective and injective objects. Then \mathcal{X} satisfies the conditions of Proposition 3.2 if and only if it is a functorially finite, resolving, and coresolving subcategory of \mathcal{A} .*

3.1 The second equivalence

The aim of this section is to present another connection between categories of morphisms and (covariant) functors. Since, in this direction, finitely presented injective functors play a significant role, we recall some known results concerning them.

Notation 3.6. *For a left Λ -module M , let*

$$- \otimes_{\Lambda} M : \text{mod-}\Lambda \rightarrow \mathcal{A}b$$

be the covariant functor sending a right Λ -module N to $N \otimes_{\Lambda} M$. For a subcategory $\mathcal{X} \subseteq \text{mod-}\Lambda^{\text{op}}$ we denote by

$$- \otimes_{\Lambda} M|_{\mathcal{X}}$$

the restriction of $- \otimes_{\Lambda} M$ to the subcategory \mathcal{X} . Similarly, we define the notations $\text{Hom}_{\Lambda}(-, M)|_{\mathcal{X}}$ and $\text{Hom}_{\Lambda}(M, -)|_{\mathcal{X}}$. For abbreviation, we usually write $(-, M)|_{\mathcal{X}}$ and $(M, -)|_{\mathcal{X}}$; moreover, as before, the symbol $|_{\mathcal{X}}$ will be omitted whenever $M \in \mathcal{X}$.

Let R be an arbitrary ring. It is proved by Auslander [3, Lemma 6.1] that for a left R -module M , the covariant functor $- \otimes_R M$ is finitely presented if and only if M is a finitely presented left R -module. It is known that there is a full and faithful functor

$$T : R\text{-mod} \longrightarrow (\text{mod-}R)\text{-mod}, \quad M \mapsto (- \otimes_R M).$$

Gruson and Jensen [12, 5.5] proved that the category $(\text{mod-}R)\text{-mod}$ has enough injectives and that the injective objects are precisely those functors isomorphic to $- \otimes_R M$ for some left R -module M .

It is natural to ask what can be said about $\text{Inj}(\mathcal{X}\text{-mod})$ when \mathcal{X} is a subcategory of a module category. In the case where R is an artin algebra Λ , we have the following result for $\text{Inj}(\mathcal{X}\text{-mod})$.

Let \mathcal{X} be a subcategory of $\text{mod-}\Lambda$ which is a dualizing k -variety. Note that $\text{mod-}\Lambda$ itself is a dualizing k -variety. Consider the following commutative diagram:

$$\begin{array}{ccc} \text{mod-}(\text{mod-}\Lambda) & \xrightarrow{D} & (\text{mod-}\Lambda)\text{-mod} \\ \downarrow |_{\mathcal{X}} & & \downarrow |_{\mathcal{X}} \\ \text{mod-}\mathcal{X} & \xrightarrow{D} & \mathcal{X}\text{-mod} \end{array}$$

where D in the rows denotes the corresponding dualities for the dualizing k -varieties $\text{mod-}\Lambda$ and \mathcal{X} , and the vertical arrows are the corresponding restriction functors.

Since the functor

$$D : \text{mod}-(\text{mod-}\Lambda) \longrightarrow (\text{mod-}\Lambda)\text{-mod}$$

is a duality, it sends projective objects to injective ones. Hence, by the result of Gruson–Jensen, we may deduce that for every $M \in \text{mod-}\Lambda$, there exists a left Λ -module M' such that

$$D(\text{Hom}_\Lambda(-, M)) \simeq - \otimes_\Lambda M'.$$

The module M' is uniquely determined up to isomorphism by the fullness and faithfulness of T .

This isomorphism restricts to the subcategory \mathcal{X} and induces

$$D(\text{Hom}_\Lambda(-, M))|_{\mathcal{X}} \simeq - \otimes_\Lambda M'|_{\mathcal{X}}.$$

If $M \in \mathcal{X}$, we may simply write

$$D((-, M)) \simeq - \otimes_\Lambda M'|_{\mathcal{X}}.$$

If Λ , considered as a right Λ -module, belongs to \mathcal{X} , then by substituting Λ into the above isomorphism we obtain $M' = D(M)$, where D is the usual duality on $\text{mod-}\Lambda$.

Therefore, we obtain the following proposition:

Proposition 3.7. *Let \mathcal{X} be a subcategory of $\text{mod-}\Lambda$ which is a dualizing k -variety and contains Λ as a right Λ -module. Then $\mathcal{X}\text{-mod}$ has enough injectives. The injective objects are precisely those functors of the form*

$$- \otimes_\Lambda M'|_{\mathcal{X}},$$

where $M' = D(X)$ for a uniquely determined (up to isomorphism) object $X \in \mathcal{X}$.

For a subcategory $\text{proj-}\Lambda \subseteq \mathcal{X} \subseteq \text{mod-}\Lambda$, denote by $\mathcal{X}^0\text{mod}$ consisting those functors of $\mathcal{X}\text{-mod}$ vanish on projective objects. Note that $\mathcal{X}^0\text{mod}$, including those functors of $\mathcal{X}\text{-mod}$ vanish on injective modules which is different of $\mathcal{X}^0\text{mod}$ that whose objects vanish on projective modules. Except in the case of when $\text{prj-}\Lambda = \text{inj-}\Lambda$, then $\mathcal{X}^0\text{mod} = \mathcal{X}\text{-mod}$.

Lemma 3.8. *Let \mathcal{X} be a subcategory of $\text{mod-}\Lambda$ and containing $\text{proj-}\Lambda$. Then for each left module $M \in \Lambda\text{-mod}$, the functor $- \otimes_\Lambda M|_{\mathcal{X}}$ becomes an object in $\mathcal{X}\text{-mod}$.*

Proof. The same argument as in [3, Lemma 6.1] works here. □

Proposition 3.9. *Assume that \mathcal{X} is a covariantly finite subcategory of $\text{mod-}\Lambda$ and that $\text{proj-}\Lambda \subseteq \mathcal{X}$. Then*

$$\mathcal{X}^0\text{mod} \simeq \underline{\mathcal{X}}\text{-mod}.$$

In particular, $\underline{\mathcal{X}}\text{-mod}$ is an abelian category with enough projectives.

Proof. The canonical functor $\pi : \mathcal{X} \rightarrow \underline{\mathcal{X}}$ induces an equivalence

$$\pi^* : \underline{\mathcal{X}}\text{-Mod} \longrightarrow \mathcal{X}^0\text{Mod},$$

where $\mathcal{X}^0\text{Mod}$ denotes the full subcategory of $\mathcal{X}\text{-Mod}$ consisting of those functors that vanish on projective modules. To obtain an equivalence between $\mathcal{X}^0\text{mod}$ and $\underline{\mathcal{X}}\text{-mod}$, it suffices to show that the restriction of π^* to $\underline{\mathcal{X}}\text{-mod}$ has its image in $\mathcal{X}^0\text{mod}$ and is dense.

Step 1: $\pi^*(F) \in \mathcal{X}^0\text{mod}$ for every $F \in \underline{\mathcal{X}}\text{-mod}$. Since π^* is exact, it is enough to check this for $F = (\underline{X}, -)$ with $X \in \mathcal{X}$. The functor $\pi^*((\underline{X}, -))$ is the same functor, now viewed on \mathcal{X} ; that is, for every $Y \in \mathcal{X}$,

$$(\underline{X}, -)(Y) \cong \underline{\text{Hom}}_{\Lambda}(X, Y) \cong \text{Hom}_{\Lambda}(X, Y) / (\text{maps factoring through projectives}).$$

By the well-known isomorphism

$$\underline{\text{Hom}}_{\Lambda}(M, -) \simeq \text{Tor}_1(-, \text{Tr}(M)) \quad (M \in \text{mod-}\Lambda),$$

we have an exact sequence

$$0 \rightarrow (\underline{X}, -) \rightarrow - \otimes_{\Lambda} \Omega(\text{Tr}(X))|_{\mathcal{X}} \rightarrow - \otimes_{\Lambda} Q|_{\mathcal{X}} \rightarrow - \otimes_{\Lambda} \text{Tr}(X)|_{\mathcal{X}} \rightarrow 0,$$

where $Q \in \text{proj } \Lambda^{\text{op}}$. Since \mathcal{X} is covariantly finite, the category $\mathcal{X}\text{-mod}$ is abelian. Thus, by the above exact sequence together with Lemma 3.8, we obtain the desired result that $\pi^*(F)$ vanishes on projective modules.

Step 2: π^* is dense. Let $F \in \mathcal{X}^0\text{mod}$, and take a projective presentation

$$(Y, -) \longrightarrow (X, -) \longrightarrow F \longrightarrow 0.$$

Since F vanishes on projective modules, this presentation fits into the commutative diagram

$$\begin{array}{ccccccc} (Y, -) & \longrightarrow & (X, -) & \longrightarrow & F & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \parallel & & \\ (\underline{Y}, -) & \longrightarrow & (\underline{X}, -) & \longrightarrow & F & \longrightarrow & 0. \end{array}$$

The lower row shows that F can be regarded as a functor on $\underline{\mathcal{X}}$, hence lies in $\underline{\mathcal{X}}\text{-mod}$. Moreover, applying π^* to this functor yields the original F viewed as a functor on \mathcal{X} . Thus, π^* is dense.

Combining Steps 1 and 2, the restriction of π^* induces an equivalence

$$\underline{\mathcal{X}}\text{-mod} \xrightarrow{\sim} \mathcal{X}^0\text{mod},$$

as required. □

In the following, we define two other functors that are important of our next theorem.

Construction 3.10. Let \mathcal{X} be a subcategory of $\text{mod-}\Lambda$, and denote by $D(\mathcal{X})$ the subcategory of all left Λ -modules of the form $D(X)$ for some $X \in \mathcal{X}$.

We define a functor

$$\Psi'_1 : \mathcal{S}_{D(\mathcal{X})}(\text{mod-}\Lambda^{\text{op}}) \longrightarrow \underline{\mathcal{X}}\text{-mod}$$

with respect to the subcategory \mathcal{X} as follows. Let $(A \xrightarrow{f} B)$ be an object in $\mathcal{S}_{D(\mathcal{X})}(\text{mod-}\Lambda^{\text{op}})$. Consider the exact sequence

$$0 \longrightarrow F \longrightarrow - \otimes_{\Lambda} A|_{\mathcal{X}} \xrightarrow{- \otimes f} - \otimes_{\Lambda} B|_{\mathcal{X}} \longrightarrow - \otimes_{\Lambda} \text{Coker}(f)|_{\mathcal{X}} \longrightarrow 0.$$

We define $\Psi'_1(A \xrightarrow{f} B) := F$. By Lemma 3.9, we can regard F as an object of $\underline{\mathcal{X}}\text{-mod}$.

Similar to Construction 2.1, the functor Ψ'_1 can also be defined on morphisms. Note that in this case we are working with injective resolutions in $\underline{\mathcal{X}}\text{-mod}$.

In a similar way, we define a functor

$$\Psi'_1 : \mathcal{F}_{D(\mathcal{X})}(\text{mod-}\Lambda^{\text{op}}) \longrightarrow \underline{\mathcal{X}}\text{-mod},$$

which sends an object $(C \xrightarrow{g} D)$ to the kernel of the morphism

$$- \otimes_{\Lambda} \ker(g)|_{\mathcal{X}} \longrightarrow - \otimes_{\Lambda} C|_{\mathcal{X}}$$

in $\underline{\mathcal{X}}\text{-mod}$, or equivalently, to a functor G fitting into the exact sequence

$$0 \longrightarrow G \longrightarrow - \otimes_{\Lambda} \ker(g)|_{\mathcal{X}} \longrightarrow - \otimes_{\Lambda} C|_{\mathcal{X}} \xrightarrow{- \otimes g} - \otimes_{\Lambda} D|_{\mathcal{X}} \longrightarrow 0.$$

Using the above construction, we are able to establish new equivalences.

Theorem 3.11. Let \mathcal{X} be a covariantly finite subcategory of $\text{mod-}\Lambda$ such that $\text{proj-}\Lambda \subseteq \mathcal{X}$ and \mathcal{X} is a dualizing k -variety. Then:

- (1) If \mathcal{X} is closed under kernels of epimorphisms, then the functor Ψ_1 , defined in Construction 3.10 with respect to \mathcal{X} , induces an equivalence of categories

$$\mathcal{S}_{D(\mathcal{X})}(\text{mod-}\Lambda^{\text{op}})/\mathcal{V}_1 \simeq \underline{\mathcal{X}}\text{-mod},$$

where

$$\mathcal{V}_1 = \text{add}\{(X \xrightarrow{\text{Id}} X) \oplus (0 \rightarrow X) \mid X \in \mathcal{X}\}.$$

- (2) If \mathcal{X} is closed under kernels of epimorphisms and cokernels of monomorphisms, then the functor Ψ'_1 , defined in Construction 3.10 with respect to \mathcal{X} , induces an equivalence of categories

$$\mathcal{F}_{D(\mathcal{X})}(\text{mod-}\Lambda^{\text{op}})/\mathcal{V}'_1 \simeq \underline{\mathcal{X}}\text{-mod},$$

where

$$\mathcal{V}'_1 = \text{add}\{(X \xrightarrow{\text{Id}} X) \oplus (X \rightarrow 0) \mid X \in \mathcal{X}\}.$$

Proof. The proofs of both statements follow the same strategy as in Theorems 2.2 and 3.1. We highlight the main points here.

Let $F \in \underline{\mathcal{X}}\text{-mod}$. By Proposition 3.9, we can view F as a functor in $\mathcal{X}\text{-mod}$ that vanishes on projectives. By Proposition 3.7, an injective resolution of F is

$$0 \longrightarrow F \longrightarrow - \otimes_{\Lambda} D(X) \xrightarrow{- \otimes D(f)} - \otimes_{\Lambda} D(Y) \longrightarrow - \otimes_{\Lambda} D(Z) \longrightarrow 0.$$

Since \mathcal{X} is closed under cokernels of monomorphisms, we have $D(Z) \in D(\mathcal{X})$. Hence, $D(X) \xrightarrow{D(f)} D(Y)$ is an object in $S_{D(\mathcal{X})}(\text{mod-}\Lambda^{\text{op}})$ with

$$\Psi_1(D(X) \xrightarrow{D(f)} D(Y)) \simeq F.$$

Thus, Ψ_1 is dense.

Regarding the kernel of Ψ_1 : any object in \mathcal{V}_1 is clearly sent to zero. Conversely, if $D(X) \xrightarrow{D(f)} D(Y)$ lies in the kernel of Ψ_1 , we have an exact sequence

$$0 \longrightarrow - \otimes_{\Lambda} D(X) \xrightarrow{- \otimes D(f)} - \otimes_{\Lambda} D(Y) \longrightarrow - \otimes_{\Lambda} D(Z) \longrightarrow 0$$

in $\mathcal{X}\text{-mod}$. Applying the duality $D = \text{Hom}_k(-, E)$ yields

$$0 \longrightarrow (-, Z) \longrightarrow (-, Y) \xrightarrow{(-, f)} (-, X) \longrightarrow 0$$

in $\text{mod-}\mathcal{X}$, showing that f is a split epimorphism, equivalently $D(f)$ is a split monomorphism. Thus, $D(f)$ is an object of \mathcal{V}_1 .

Finally, the fullness and exactness of Ψ_1 can be shown as in the proof of Theorem 2.2, with the difference that we now work with injective resolutions instead of projective resolutions.

For part (2), the additional assumption that \mathcal{X} is closed under kernels of epimorphisms is needed to ensure that Ψ'_1 is dense. \square

The difference between the equivalences in Theorems 2.2 and 3.1 and those in the above theorem is that, in the latter, left and right modules are simultaneously involved.

4 Different Exact Structures on the Monomorphism Categories

Let \mathcal{X} be a resolving and contravariantly finite subcategory of $\text{mod-}\Lambda$. In Subsection 2.1, we assign to \mathcal{X} the subcategory $\mathcal{S}_{\mathcal{X}}(\Lambda)$ of the morphism category $\text{H}(\Lambda)$, consisting of all monomorphisms $(A \xrightarrow{f} B)$ with A , B , and $\text{Cok } f$ in \mathcal{X} . Since $\mathcal{S}_{\mathcal{X}}(\Lambda)$ is closed under extensions, it gets a natural exact structure from $\text{H}(\Lambda)$. In [22], we introduce two other exact structures on $\mathcal{S}_{\mathcal{X}}(\Lambda)$, distinct from the canonical one, and completely classify the indecomposable projective (resp. injective) objects in the corresponding exact categories.

Equipping $\mathcal{S}_{\mathcal{X}}(\Lambda)$ with one of these new exact structures provides a framework for constructing a triangle functor. We next use this functor to establish a triangle equivalence between the bounded derived category $\mathbb{D}^b(\text{mod-}\mathcal{X})$ and a Verdier quotient of the bounded derived category of the associated exact category on $\mathcal{S}_{\mathcal{X}}(\Lambda)$. A similar analysis is also carried out for the singularity category of $\text{mod-}\mathcal{X}$. In fact, we obtain derived or singularity versions of the equivalences that appear in the context of monomorphism categories, for instance, in [35, 7, 15].

4.1 Three types of exact categories

In the paper [22], the authors study three different kinds of exact structures on $\mathcal{S}_{\mathcal{X}}(\Lambda)$ for a contravariantly finite resolving subcategory \mathcal{X} of $\text{mod-}\Lambda$ as follows and then they characterize the indecomposable projective and injective objects for each type of the exact categories, and moreover see whether or not these exact categories have enough projectives and injectives. The characterization of projective, or injective, objects in three types of the exact categories over $\mathcal{S}_{\mathcal{X}}(\Lambda)$ implies that they are different as exact categories, whenever Λ is assumed not to be semisimple.

As an application of the classification of projective objects and injective objects one can see that the exact category $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\mathcal{X})$ is Frobenius whenever the canonical exact category \mathcal{X} is so. Let us state this observation as the following result.

- **Case 1:** Since \mathcal{X} is closed under extensions, $\mathcal{S}_{\mathcal{X}}(\Lambda)$ is closed under extensions in the morphism category $\text{H}(\Lambda)$. Hence, $\mathcal{S}_{\mathcal{X}}(\Lambda)$ becomes naturally an exact category by the class of all short exact sequences in $\text{H}(\Lambda)$ with terms in $\mathcal{S}_{\mathcal{X}}(\Lambda)$. We denote by $\mathcal{E}_{\mathcal{X}}$ such short exact sequences.
- **Case 2:** Let $\mathcal{E}_{\mathcal{X}}^{\text{cw}} \subseteq \mathcal{E}_{\mathcal{X}}$ be the class of all exact sequences

$$0 \longrightarrow \begin{array}{ccc} X_1 & & Z_1 \\ \downarrow f & \xrightarrow{\phi_1} & \downarrow h \\ X_2 & \xrightarrow{\phi_2} & Z_2 \end{array} \xrightarrow{\psi_1} \begin{array}{ccc} Y_1 & & \\ \downarrow g & & \\ Y_2 & & \end{array} \longrightarrow 0$$

such that the sequences $0 \rightarrow X_i \xrightarrow{\phi_i} Z_i \xrightarrow{\psi_i} Y_i \rightarrow 0$, $i = 1, 2$, in \mathcal{X} split.

- **Case 3:** Let

$$\epsilon : 0 \longrightarrow \begin{array}{ccc} X_1 & & Z_1 \\ \downarrow f & \xrightarrow{\phi_1} & \downarrow h \\ X_2 & \xrightarrow{\phi_2} & Z_2 \end{array} \xrightarrow{\psi_1} \begin{array}{ccc} Y_1 & & \\ \downarrow g & & \\ Y_2 & & \end{array} \longrightarrow 0$$

be a short exact sequence in $\mathcal{E}_{\mathcal{X}}$. Expanding the above sequence in $\text{mod-}\Lambda$, we

obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & (*) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X_1 & \xrightarrow{\phi_1} & Z_1 & \xrightarrow{\psi_1} & Y_1 & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow h & & \downarrow g & & \\
 0 & \longrightarrow & X_2 & \xrightarrow{\phi_2} & Z_2 & \xrightarrow{\psi_2} & Y_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Cok } f & \xrightarrow{\mu_1} & \text{Cok } h & \xrightarrow{\mu_2} & \text{Cok } g & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

in $\text{mod-}\Lambda$. Denote by $\mathcal{E}_{\mathcal{X}}^{\text{scw}}$, the class of all those short exact sequences ϵ in $\mathcal{E}_{\mathcal{X}}$ such that all the rows in their induced diagram $(*)$ in $\text{mod-}\Lambda$ are split. Clearly, $\mathcal{E}_{\mathcal{X}}^{\text{scw}} \subseteq \mathcal{E}_{\mathcal{X}}^{\text{cw}}$.

Let $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ denote the exact category $(\mathcal{S}_{\mathcal{X}}(\Lambda), \mathcal{E}_{\mathcal{X}}^{\text{scw}})$. For the case $\mathcal{X} = \text{mod-}\Lambda$, we usually write only $\mathcal{S}^{\text{scw}}(\Lambda)$. Since for our next purpose we mainly need the exact structure of Case 3, we provide more details about this type of exact structure in what follows. A complete characterization of indecomposable projective and injective objects is given below.

Let M be a module in $\text{mod-}\Lambda$ and $P_M \xrightarrow{\pi_M} M$ the projective cover of M in $\text{mod-}\Lambda$. Set $\Omega_{\Lambda}(M)$ the kernel of π_M and $\Omega_{\Lambda}(M) \hookrightarrow P_M$ the canonical inclusion.

We should mention that since \mathcal{X} is closed under extensions it inherits an exact structure from $\text{mod-}\Lambda$ by the class of all short exact sequences in $\text{mod-}\Lambda$ with the terms in \mathcal{X} . In the sequel, we view \mathcal{X} as an exact category with this inherited exact structure.

Proposition 4.1. *Let $(X_1 \xrightarrow{f} X_2)$ be an object in $\mathcal{S}_{\mathcal{X}}(\Lambda)$. Then f is an indecomposable projective object in the exact category $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ if and only if it is isomorphic to either of the following objects*

- (1) $(X \xrightarrow{1} X)$ for some indecomposable object X in \mathcal{X} ;
- (2) $(0 \rightarrow X)$ for some indecomposable object X in \mathcal{X} ;
- (3) $(\Omega_{\Lambda}(X) \hookrightarrow P_X)$ for some indecomposable object X in \mathcal{X} .

Moreover, $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ has enough projectives.

Proposition 4.2. *Suppose that the exact category \mathcal{X} has enough injectives. Let $(X_1 \xrightarrow{f} X_2)$ be an object in $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$. Then f is an indecomposable injective object in the exact category $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ if and only if it is isomorphic to either of the following objects*

- (1) $(X \xrightarrow{1} X)$ for some indecomposable object X in \mathcal{X} ;

- (2) $(X \xrightarrow{l} I)$ for some indecomposable object X in \mathcal{X} , where l is a left minimal monomorphism, i.e., any endomorphism g with $g \circ l = l$ is an automorphism, and I an injective object in the exact category \mathcal{X} ;
- (3) $(0 \rightarrow I)$ for some indecomposable injective object I in the exact category \mathcal{X} .

Corollary 4.3. *Assume that the exact category \mathcal{X} has enough injective. If the exact category \mathcal{X} is Frobenius, then so is $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$.*

The subcategory of Gorenstein projective modules over an Iwanaga-Gorenstein algebra satisfies the assumption of the above corollary.

4.2 Equivalences on the stable categories

Recall from Construction 2.1, the functor $\Psi_{\mathcal{X}} : \mathcal{S}_{\mathcal{X}}(\Lambda) \rightarrow \text{mod-}\underline{\mathcal{X}}$ which is assigned to the \mathcal{X} making a connection between the monomorphism category and modules (or functors) over the stable categories. Since $\text{mod-}\underline{\mathcal{X}}$ is an abelian category, it is reasonable to ask how $\Psi_{\mathcal{X}}$ becomes an exact functor. With the first and second types of the exact structures defined on $\mathcal{S}_{\mathcal{X}}(\Lambda)$ in the previous section, the functor $\Psi_{\mathcal{X}}$ is not an exact functor in general.

For instance, let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a non-split short exact sequence in \mathcal{X} . Then the functor $\Psi_{\mathcal{X}}$ does not preserve the exactness of the following short exact sequence in $\mathcal{S}_{\mathcal{X}}(\Lambda)$

$$0 \longrightarrow \begin{array}{c} A \\ \downarrow 1 \\ A \end{array} \xrightarrow{f} \begin{array}{c} A \\ \downarrow f \\ B \end{array} \xrightarrow{g} \begin{array}{c} 0 \\ \downarrow \\ C \end{array} \longrightarrow 0.$$

For the second type, assume the projective dimensions of A and B are one. The Horseshoe lemma implies the following commutative diagram in $\text{mod-}\Lambda$

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q_1 & \longrightarrow & Q_1 \oplus P_1 & \longrightarrow & P_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q_2 & \longrightarrow & Q_2 \oplus P_2 & \longrightarrow & P_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where P_i, Q_i are projective. The top two rows give us a short exact sequence in $\mathcal{E}_{\mathcal{X}}^{\text{cw}}$ so that its image under $\Psi_{\mathcal{X}}$ can not be exact. Hence, $\Psi_{\mathcal{X}} : \mathcal{S}_{\mathcal{X}}^{\text{cw}}(\Lambda) \rightarrow \text{mod-}\underline{\mathcal{X}}$ is not in general an exact functor as well. But, by considering the third type of exact structures on $\mathcal{S}_{\mathcal{X}}(\Lambda)$, the functor turns into an exact functor. Therefore, $\Psi_{\mathcal{X}} : \mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda) \rightarrow \text{mod-}\underline{\mathcal{X}}$ becomes an exact functor from the exact category $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ to $\text{mod-}\underline{\mathcal{X}}$.

By enhancing $\Psi_{\mathcal{X}}$ with such an exactness, we can reformulate Theorem 2.2 as follows:

Theorem 4.4. *Let \mathcal{X} be a contravariantly finite resolving subcategory of $\text{mod-}\Lambda$. Then*

- (1) *The functor $\Psi_{\mathcal{X}} : \mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda) \rightarrow \text{mod-}\underline{\mathcal{X}}$ is full, dense, objective and exact;*
- (2) *The functor $\Psi_{\mathcal{X}}$ induces an equivalence between the stable category $\underline{\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)}$ and $\text{mod-}\underline{\mathcal{X}}$.*

We can give another variation of the equivalence $\mathcal{S}_{\mathcal{X}}(\Lambda)/\mathcal{U} \simeq \text{mod-}\underline{\mathcal{X}}$ on the level of the stable categories as in the following.

Theorem 4.5. *Keep all the notations as above. The functor $\Psi_{\mathcal{X}}$ induces an equivalence between the stable categories of exact categories $\underline{\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)} \simeq \underline{\text{mod-}\mathcal{X}}$, (the induced equivalence denoted by $\underline{\Psi}_{\mathcal{X}}$). Consequently, there is an equivalence of categories*

$$\mathcal{S}_{\mathcal{X}}(\Lambda)/\mathcal{V} \simeq \underline{\text{mod-}\mathcal{X}},$$

where \mathcal{V} is an ideal generated by the object of the form $(X \xrightarrow{1} X)$, $(0 \rightarrow X)$ and $(\Omega_{\Lambda}(X) \rightarrow P)$, for some X in \mathcal{X} .

In Corollary 4.3, we saw when the exact category \mathcal{X} is Frobenius then the exact category $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ becomes a Frobenius exact category. Assume the exact category \mathcal{X} is Frobenius, in view of [26, Chapter I, Section 2], the stable category $\underline{\mathcal{X}}$ and $\underline{\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)}$ get naturally a triangulated structure. On the other hand, since $\underline{\mathcal{X}}$ is a triangulated category, by a classical result from [11, Theorem 3.1], $\text{mod-}\underline{\mathcal{X}}$ is a Frobenius (abelian) category. Consequently, the stable category $\underline{\text{mod-}\mathcal{X}}$ similarly gets a triangulated structure in a natural way. Hence, in this case the functor $\underline{\Psi}_{\mathcal{X}}$ turns out to be a functor between triangulated categories. The natural question may arise here whether $\underline{\Psi}_{\mathcal{X}}$ is a triangulated functor. In the sequel, we will show that the answer to this question is in the affirmative.

Theorem 4.6. *Keep all the notations as above. Assume the exact category \mathcal{X} is Frobenius. The functor $\Psi_{\mathcal{X}}$ induces a triangle equivalence between the triangulated categories $\underline{\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)} \simeq \underline{\text{mod-}\mathcal{X}}$.*

4.3 Equivalence of singularity categories

In Theorem 4.4, we observed that $\Psi_{\mathcal{X}}$ becomes an exact functor $\Psi_{\mathcal{X}} : \mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda) \rightarrow \text{mod-}\underline{\mathcal{X}}$. Therefore, $\Psi_{\mathcal{X}}$ induces a triangle functor

$$\mathbb{D}^b\Psi_{\mathcal{X}} : \mathbb{D}^b(\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)) \rightarrow \mathbb{D}^b(\text{mod-}\underline{\mathcal{X}})$$

between the derived category $\mathbb{D}^b(\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda))$ of the exact category $\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)$ and the derived category $\mathbb{D}^b(\text{mod-}\underline{\mathcal{X}})$ of the abelian category $\text{mod-}\underline{\mathcal{X}}$. We will show this functor induces the triangulated equivalences. Then, we will give an interesting application for singular equivalences of Morita type.

Theorem 4.7. [22, Theorems 6.5, 6.6] *Let \mathcal{X} be a contravariantly finite resolving subcategory \mathcal{X} of $\text{mod-}\Lambda$. Let \mathcal{U} denote the smallest thick subcategory of $\mathbb{D}^b(\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda))$ containing all complexes concentrated in degree zero with terms of the forms $(X \xrightarrow{1} X)$ and $(0 \rightarrow X)$, where X runs through all objects in \mathcal{X} . Then there exists the following equivalences of triangulated categories*

$$\begin{aligned}\mathbb{D}^b(\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda))/\mathcal{U} &\simeq \mathbb{D}^b(\text{mod-}\underline{\mathcal{X}}), \\ \mathbb{D}_{\text{sg}}(\mathcal{S}_{\mathcal{X}}^{\text{scw}}(\Lambda)) &\simeq \mathbb{D}_{\text{sg}}(\text{mod-}\underline{\mathcal{X}}).\end{aligned}$$

If we apply the above theorem for the subcategory of Gorenstein projective modules, we obtain the following corollary.

Corollary 4.8. *Assume that Λ is a Gorenstein algebra. Then, there is the following equivalence of triangulated categories*

$$\mathbb{D}_{\text{sg}}(\mathcal{S}_{\text{mod-}\Lambda}^{\text{scw}}(\Lambda)) \simeq \underline{\mathcal{S}_{\text{Gprj-}\Lambda}^{\text{scw}}(\Lambda)}.$$

We end this section with some application for singular equivalences of Morita type. Recently Xiao-Wu Chen and Long-Gang Sun introduced singular equivalences of Morita type [5], see also [38].

Definition 4.9. *Let A and B be two finite dimensional algebras. We say that A and B are singularly equivalent of Morita type if there is a pair of bimodules $({}_A M_B, {}_B N_A)$ satisfying the following conditions:*

- (1) M is finitely generated and projective as A^{op} -module and as B -module;
- (2) N is finitely generated and projective as A -module and as B^{op} -module;
- (3) There is a finitely generated A - A -bimodule X with finite projective dimension such that ${}_A M \otimes_B {}_B N_A \simeq {}_A A_A \oplus {}_A X_A$;
- (4) There is a finitely generated B - B -bimodule Y with finite projective dimension such that ${}_B N \otimes_A {}_A M_B \simeq {}_B B_B \oplus {}_B Y_B$.

A direct consequence of the above definition is that $- \otimes_A M_B : \mathbb{D}_{\text{sg}}(A) \rightarrow \mathbb{D}_{\text{sg}}(B)$ is an equivalence of triangulated categories with quasi-inverse $- \otimes_B N_A : \mathbb{D}_{\text{sg}}(B) \rightarrow \mathbb{D}_{\text{sg}}(A)$, see also [38, Proposition 2.3].

As an application of Theorem 4.7, we obtain the following result, which extends a singular equivalence of Morita type to a triangulated equivalence between the corresponding singularity categories.

Theorem 4.10. [22, Theorem 9] *Let A and B be two finite dimensional algebras. If A and B are singularly equivalent of Morita type, then there is the following equivalence of triangulated categories*

$$\mathbb{D}_{\text{sg}}(\text{mod-}\underline{\text{mod-}}A) \simeq \mathbb{D}_{\text{sg}}(\text{mod-}\underline{\text{mod-}}B).$$

5 On the Monomorphism Category of n -Cluster Tilting Subcategories

An additive subcategory \mathcal{M} of $\text{mod-}\Lambda$ is called an n -cluster tilting subcategory if it is a functorially finite and generating-cogenerating subcategory of $\text{mod-}\Lambda$ such that $\mathcal{M}^{\perp n} = \mathcal{M} = {}^{\perp n}\mathcal{M}$, where

$$\mathcal{M}^{\perp n} := \{A \in \text{mod-}\Lambda \mid \text{Ext}_{\Lambda}^i(\mathcal{M}, A) = 0 \text{ for all } 0 < i < n\},$$

$${}^{\perp n}\mathcal{M} := \{A \in \text{mod-}\Lambda \mid \text{Ext}_{\Lambda}^i(A, \mathcal{M}) = 0 \text{ for all } 0 < i < n\}.$$

Let \mathcal{M} be an n -cluster tilting subcategory of $\text{mod-}\Lambda$. Let $\mathcal{S}(\mathcal{M})$ denote the full subcategory of $\mathcal{S}(\Lambda)$, the submodule category of Λ , consisting of all monomorphisms in \mathcal{M} . In [2], we construct two functors from $\mathcal{S}(\mathcal{M})$ to $\text{mod-}\underline{\mathcal{M}}$, the category of finitely presented (coherent) additive contravariant functors on the stable category of \mathcal{M} . These functors can be viewed as a higher version of the two functors investigated by Ringel and Zhang [35] in the case $\Lambda = k[t]/\langle t^n \rangle$ and extended later by Eiríksson [7] to self-injective artin algebras.

5.1 The functor $\Phi : \mathcal{S}(\mathcal{M}) \longrightarrow \text{mod-}\underline{\mathcal{M}}$

Let Λ be an artin algebra and \mathcal{M} be an n -cluster tilting subcategory of $\text{mod-}\Lambda$, where $n > 1$ is a fixed positive integer. To define Φ we require some preparations. In particular, we need to define two functors Υ and i_{λ} . First we need the following notations:

- Let $\tilde{\mathcal{P}}^{\leq 1}(\mathcal{M})$ denote the subcategory of $\text{mod-}\underline{\mathcal{M}}$ consisting of all finitely presented functors of projective dimension at most one.
- Consider the subcategory

$$\mathcal{S}(\mathcal{M}) := \{M_1 \xrightarrow{f} M_2 \mid f \in \mathcal{S}(\Lambda) \text{ and } M_1, M_2 \in \mathcal{M}\},$$

of the morphism category $\mathcal{S}(\Lambda)$.

The assignment

$$(M_1 \xrightarrow{f} M_2) \mapsto \text{Coker}(\mathcal{M}(-, M_1) \xrightarrow{\mathcal{M}(-, f)} \mathcal{M}(-, M_2))$$

defines a functor

$$\Upsilon : \mathcal{S}(\mathcal{M}) \longrightarrow \tilde{\mathcal{P}}^{\leq 1}(\mathcal{M}).$$

By applying [1, Theorem 3.5] for the n -cluster tilting subcategory \mathcal{M} , we have a recollement

$$\begin{array}{ccccc} \text{mod}_0\text{-}\mathcal{M} & \begin{array}{c} \xleftarrow{i_{\lambda}} \\ \xrightarrow{i} \\ \xleftarrow{i_{\rho}} \end{array} & \text{mod-}\mathcal{M} & \begin{array}{c} \xleftarrow{\vartheta_{\lambda}} \\ \xrightarrow{\vartheta} \\ \xleftarrow{\vartheta_{\rho}} \end{array} & \text{mod-}\Lambda, \end{array}$$

in which $\text{mod}_0\text{-}\mathcal{M} := \text{Ker}\vartheta$ is the full subcategory of $\text{mod-}\mathcal{M}$ consisting of all functors F such that $\vartheta(F) = 0$, equivalently, $\text{mod}_0\text{-}\mathcal{M}$ consists of all functors $F \in \text{mod-}\mathcal{M}$ that vanish on $\text{prj-}\Lambda$, the full subcategory of $\text{mod-}\Lambda$ of all projective Λ -modules.

Now we have the necessary material to introduce the functor Φ . Consider the composition

$$\mathcal{S}(\mathcal{M}) \xrightarrow{\Upsilon} \tilde{\mathcal{P}}^{\leq 1}(\mathcal{M}) \xrightarrow{i_\lambda|} \text{mod-}\underline{\mathcal{M}}$$

and set $\Phi := i_\lambda| \circ \Upsilon$.

Theorem 5.1. [2, Theorem 3.5 and Corollary 3.6] *Let Λ be a self-injective artin algebra. The functor $\Phi : \mathcal{S}(\mathcal{M}) \rightarrow \text{mod-}\underline{\mathcal{M}}$ is full, dense and objective. In particular, there exists an equivalence of abelian categories*

$$\mathcal{S}(\mathcal{M})/\mathcal{U} \simeq \text{mod-}\underline{\mathcal{M}},$$

where \mathcal{U} is the subcategory of $\mathcal{S}(\mathcal{M})$ generated by the objects of the form $(M \xrightarrow{1} M)$ and $(M \xrightarrow{f} P)$, where $M \in \mathcal{M}$ and $P \in \text{prj-}\Lambda$.

5.2 The functor $\Psi : \mathcal{S}(\mathcal{M}) \rightarrow \text{mod-}\underline{\mathcal{M}}$

Here, we introduce the second functor. It can be viewed a higher version of the functor in Subsection 2.1.

Let $M_1 \xrightarrow{f} M_2$ be an object of $\mathcal{S}(\mathcal{M})$. Since \mathcal{M} is an n -cluster tilting subcategory, we may take an n -cokernel of f which leads to an n -exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{d^1} M^1 \xrightarrow{d^2} M^2 \longrightarrow \dots \longrightarrow M^{n-1} \xrightarrow{d^n} M^n \longrightarrow 0$$

Hence, the following induced sequence

$$0 \rightarrow \mathcal{M}(-, M_1) \rightarrow \mathcal{M}(-, M_2) \rightarrow \mathcal{M}(-, M^1) \rightarrow \dots \rightarrow \mathcal{M}(-, M^n) \rightarrow F \rightarrow 0$$

is exact, where F is the cokernel of the morphism $\mathcal{M}(-, M^{n-1}) \rightarrow \mathcal{M}(-, M^n)$. Clearly F vanishes on projective modules and so $F \in \text{mod-}\underline{\mathcal{M}}$. We define a functor

$$\Psi : \mathcal{S}(\mathcal{M}) \longrightarrow \text{mod-}\underline{\mathcal{M}}$$

by setting $\Psi(M_1 \xrightarrow{f} M_2) = F$. First of all, since every two n -cokernels of f are homotopy equivalent, we deduce that the definition of Ψ is independent of the choice of the n -cokernel of F .

Now let $f : M_1 \rightarrow M_2$ and $f' : M'_1 \rightarrow M'_2$ be two objects of $\mathcal{S}(\mathcal{M})$ and consider a morphism $f \xrightarrow{\alpha_1}{\alpha_2} f'$. By the property of n -exact sequences, we infer that (α_1, α_2) lifts to the following morphism of n -exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{f} & M_2 & \xrightarrow{d^1} & M^1 & \longrightarrow & \dots & \longrightarrow & M^{n-1} & \xrightarrow{d^n} & M^n & \longrightarrow & 0 \\ & & \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha^1 & & & & \downarrow \alpha^{n-1} & & \downarrow \alpha^n & & \\ 0 & \longrightarrow & M'_1 & \xrightarrow{f'} & M'_2 & \xrightarrow{d'^1} & M'^1 & \longrightarrow & \dots & \longrightarrow & M'^{n-1} & \xrightarrow{d'^n} & M'^n & \longrightarrow & 0. \end{array} \quad (1)$$

Yoneda's lemma now comes to play to induce the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{M}(-, M_1) & \longrightarrow & \mathcal{M}(-, M_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{M}(-, M^n) & \longrightarrow & F & \longrightarrow & 0 \\
& & \downarrow \mathcal{M}(-, \alpha_1) & & \downarrow \mathcal{M}(-, \alpha_2) & & & & \downarrow \mathcal{M}(-, \alpha^n) & & \downarrow \eta & & \\
0 & \longrightarrow & \mathcal{M}(-, M'_1) & \longrightarrow & \mathcal{M}(-, M'_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{M}(-, M'^n) & \longrightarrow & F' & \longrightarrow & 0.
\end{array}$$

We set $\Psi(\alpha_1, \alpha_2) = \eta$. The comparison Lemma follows that η is independent of the lifting morphism $\{\alpha^j\}_{1 \leq j \leq n}$.

Theorem 5.2. [2, Theorem 4.1 and Corollary 4.2] *The functor $\Psi : \mathcal{S}(\mathcal{M}) \longrightarrow \text{mod-}\underline{\mathcal{M}}$ is full, dense and objective. In particular, there exists an equivalence of additive categories*

$$\mathcal{S}(\mathcal{M})/\mathcal{V} \simeq \text{mod-}\underline{\mathcal{M}},$$

where \mathcal{V} is the full subcategory of $\mathcal{S}(\mathcal{M})$ generated by all finite direct sums of objects of the form $(M \xrightarrow{1} M)$ and $(0 \rightarrow M)$, where M runs over objects of \mathcal{M} .

5.3 Comparison

In the rest, we compare the functors Ψ and Φ . Such a comparison is motivated by [35, Theorem 2] and [7, Theorem 4.2]. Let $\Omega_{\underline{\mathcal{M}}} : \underline{\text{mod-}}\underline{\mathcal{M}} \longrightarrow \underline{\text{mod-}}\underline{\mathcal{M}}$ denote the syzygy functor. Note that since $\underline{\text{mod-}}\underline{\mathcal{M}}$ is semi-perfect, we can assume that $\Omega_{\underline{\mathcal{M}}}(F)$ is the kernel of a projective cover of F in $\underline{\text{mod-}}\underline{\mathcal{M}}$.

Let \mathcal{W} denote the smallest additive subcategory of $\mathcal{S}(\mathcal{M})$ generated by all objects of the form $M \xrightarrow{1} M$, $0 \rightarrow M$ and $M \rightarrow P$, with $P \in \text{prj-}\Lambda$. Since Φ sends \mathcal{W} to zero, there exists an induced functor $\mathcal{S}(\mathcal{M})/\mathcal{W} \longrightarrow \underline{\text{mod-}}\underline{\mathcal{M}}$ that will be denoted by $\underline{\Phi}$.

The concept of $n\mathbb{Z}$ -cluster tilting subcategories is given by Iyama and Jasso [29], as subcategories that are closed under n -syzygies and n -cosyzygies and so behave well from the viewpoint of higher homological algebra. We say that an n -cluster tilting subcategory \mathcal{M} of $\text{mod-}\Lambda$ is $n\mathbb{Z}$ -cluster tilting if it admits n -syzygies, i.e. $\Omega^n(\mathcal{M}) \subseteq \mathcal{M}$ or equivalently, if it admits n -cosyzygies, i.e. $\Omega^{-n}(\mathcal{M}) \subseteq \mathcal{M}$. See [29, Definition-Proposition 2.15] for more equivalent statements.

Now assume that \mathcal{M} is an $n\mathbb{Z}$ -cluster tilting subcategory. Then, by definition, $\Psi(0 \rightarrow M) = 0$, $\Psi(M \rightarrow M) = 0$, and using the fact that Λ is self-injective, $\Psi(M \rightarrow P) = \underline{\mathcal{M}}(-, \Omega_{\Lambda}^{-n}(M))$. Hence, in this case also we obtain the induced functor $\underline{\Psi} : \mathcal{S}(\mathcal{M})/\mathcal{W} \rightarrow \underline{\text{mod-}}\underline{\mathcal{M}}$.

Theorem 5.3. *Let Λ be a self-injective artin algebra and \mathcal{M} be an $n\mathbb{Z}$ -cluster tilting subcategory of $\text{mod-}\Lambda$. Then, with the above notations, we have*

$$\underline{\Phi} = \Omega_{\underline{\mathcal{M}}}^n \circ \underline{\Psi}.$$

That is, the functors $\underline{\Phi}$ and $\underline{\Psi}$ differ by the n -syzygy functor on $\underline{\text{mod-}}\underline{\mathcal{M}}$.

As an application of our result in the higher setting, we prove a higher version of Hilton-Rees theorem for n -cluster tilting subcategories.

Theorem 5.4. (*Higher Hilton-Rees*) Let \mathcal{M} be an n -cluster tilting subcategory of $\text{mod-}\Lambda$, and X, Y in \mathcal{M} .

- (i) There is an isomorphism between $\underline{\mathcal{M}}(X, Y)$ and the group of natural transformations from $\text{Ext}_{\Lambda}^n(X, -)|_{\mathcal{M}}$ to $\text{Ext}_{\Lambda}^n(Y, -)|_{\mathcal{M}}$.
- (ii) There is an isomorphism between $\overline{\mathcal{M}}(X, Y)$ and the group of natural transformations from $\text{Ext}_{\Lambda}^n(-, X)|_{\mathcal{M}}$ to $\text{Ext}_{\Lambda}^n(-, Y)|_{\mathcal{M}}$.

Our next purpose is to state and prove a higher version of Auslander's direct summand conjecture [3] for n -cluster tilting subcategories of $\text{mod-}\Lambda$. The conjecture says that for an object A of an abelian category \mathcal{A} with enough projective objects, any direct summand F of $\text{Ext}_{\mathcal{A}}^1(A, -)$ is of the form $\text{Ext}_{\mathcal{A}}^1(B, -)$, for some B in \mathcal{A} . For a review of the conjecture and relevant results, see the introduction of [33].

Theorem 5.5. Let \mathcal{M} be an n -cluster tilting subcategory of $\text{mod-}\Lambda$.

- (i) If F is a direct summand of $\text{Ext}_{\Lambda}^n(A, -)|_{\mathcal{M}}$, then there exists $B \in \mathcal{M}$ such that $F \simeq \text{Ext}_{\Lambda}^n(B, -)|_{\mathcal{M}}$.
- (ii) If F is a direct summand of $\text{Ext}_{\Lambda}^n(-, A)|_{\mathcal{M}}$, then there exists $B \in \mathcal{M}$ such that $F \simeq \text{Ext}_{\Lambda}^n(-, B)|_{\mathcal{M}}$.

At the end of this paper, we re-proves the existence of n -Auslander-Reiten translation $\tau_n = \tau\Omega_{\Lambda}^{n-1}$ that is already known by [28]. Our method provides a functorial approach for the existence of n -Auslander-Reiten translation.

6 The stable Auslander-Reiten components of certain monomorphism categories

Let Λ be an Artin algebra and let $\text{Gprj-}\Lambda$ denote the class of all finitely generated Gorenstein projective Λ -modules. Following Subsection 2.1, we can define the monomorphism category $\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)$. In [25], this category is denoted by $\mathcal{S}_{\mathcal{G}}(\Lambda)$. There, we study the components of the stable Auslander-Reiten quiver of $\mathcal{S}_{\mathcal{G}}(\Lambda)$ and describe their shapes. It is shown that certain components are linked to the orbits of an auto-equivalence on the stable category $\underline{\text{Gprj-}}\Lambda$. In particular, for the finite components, we prove that under certain mild conditions their cardinalities are divisible by 3.

Throughout, let $\text{Gprj-}\Lambda$ be a contravariantly finite subcategory of $\text{mod-}\Lambda$. It is proved in [18, Lemma 5.2] that $\mathcal{S}_{\mathcal{G}}(\Lambda)$ is a functorially finite subcategory of $\text{H}(\Lambda)$. Therefore, $\mathcal{S}_{\mathcal{G}}(\Lambda)$ has almost split sequences, in particular if Λ is Gorenstein or a CM-finite algebra.

The following lemma determines the structure of certain almost split sequences in $\mathcal{S}_{\mathcal{G}}(\Lambda)$. This lemma plays a crucial role in this paper.

Lemma 6.1. ([15, Lemma 6.3]) Let $\delta : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an almost split sequence in $\text{Gprj-}\Lambda$.

(1) The almost split sequence in $\mathcal{S}_G(\Lambda)$ ending at $(0 \rightarrow C)$ has the form

$$0 \longrightarrow (A \xrightarrow{1} A) \xrightarrow{(1 \ f)} (A \xrightarrow{f} B) \xrightarrow{(0 \ g)} (0 \rightarrow C) \longrightarrow 0. \quad (2)$$

(2) Let $e : A \rightarrow I$ be the minimal left $\text{prj-}\Lambda$ -approximation of A . Then the almost split sequence in $\mathcal{S}_G(\Lambda)$ ending at $(C \xrightarrow{1} C)$ has the form

$$0 \longrightarrow (A \xrightarrow{e} I) \xrightarrow{(f \ \begin{bmatrix} 1 \\ 0 \end{bmatrix})} (B \xrightarrow{h} I \oplus C) \xrightarrow{(g \ [0 \ 1])} (C \xrightarrow{1} C) \longrightarrow 0, \quad (3)$$

where $e' : B \rightarrow I$ is an extension of e and h is the map $\begin{bmatrix} e' \\ g \end{bmatrix}$, that is, h is induced by the following push-out diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow e & & \downarrow h & & \parallel \\ I & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & I \oplus C & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & C \end{array}$$

(3) Let $b : P \rightarrow C$ be the projective cover of C . Then the almost split sequence in $\mathcal{S}_G(\Lambda)$ starting with $(0 \rightarrow A)$ has the form

$$0 \longrightarrow (0 \rightarrow A) \xrightarrow{(0 \ \begin{bmatrix} 1 \\ 0 \end{bmatrix})} (\Omega_\Lambda(C) \xrightarrow{h} A \oplus P) \xrightarrow{(1 \ [0 \ 1])} (\Omega_\Lambda(C) \xrightarrow{i} P) \longrightarrow 0, \quad (4)$$

where b' is a lifting of b to g and h is the kernel of the morphism $\begin{bmatrix} f & b' \end{bmatrix} : A \oplus P \rightarrow B$, that is, h is induced by the following pull-back diagram

$$\begin{array}{ccccc} & & \Omega_\Lambda(C) & \xlongequal{\quad} & \Omega_\Lambda(C) \\ & & \downarrow h & & \downarrow i \\ A & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & A \oplus P & \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} & P \\ \parallel & & \downarrow \begin{bmatrix} f & b' \end{bmatrix} & & \downarrow b \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

If we start with an indecomposable non-projective object $(0 \rightarrow C)$, then after three applications of the lemma we obtain the indecomposable

$$(0 \rightarrow \tau_G \Omega_\Lambda^{-1} \tau_G^2 C).$$

This is the first place where we observe 3-periodicity, meaning that after three steps we arrive at an object of the same form as the one we started with, namely an object with zero domain. Motivated by this phenomenon, we let ϑ denote the composition

$$\vartheta := \tau_G \Omega_\Lambda^{-1} \tau_G^2,$$

where τ_G denotes the Auslander–Reiten translation in $\text{Gprj-}\Lambda$. Note that ϑ is an auto-equivalence on $\text{Gprj-}\Lambda$.

Here we should point out that the middle terms of the almost split sequences in (2) and (3) were studied in [25, Propositions 3.9, 3.10]. It was proved that each middle term is either indecomposable, or a direct sum of an indecomposable non-projective object and some copies of projective objects.

6.1 Stable Auslander-Reiten components

Let $\Gamma_{\mathcal{S}_G(\Lambda)}^s$ be the stable Auslander-Reiten quiver of $\Gamma_{\mathcal{S}_G(\Lambda)}$. For the ease of notation, $\Gamma_{\mathcal{S}_G(\Lambda)}$ and $\Gamma_{\mathcal{S}_G(\Lambda)}^s$ will be denoted by $\Gamma_{\mathcal{S}}$ and $\Gamma_{\mathcal{S}}^s$, respectively.

Definition 6.2. *A vertex in $\Gamma_{\mathcal{S}}^s$ is said to be a boundary vertex if it has one of the forms*

$$(a) (0 \rightarrow G); \quad (b) (G \xrightarrow{1} G) \quad \text{or} \quad (c) (\Omega_{\Lambda}(G) \xrightarrow{i} P),$$

where G is an indecomposable Gorenstein projective module.

Note that the boundary vertices here are a very substantial generalization of the boundary modules in [36, (5.1)].

For our next theorem we need to recall the definition of the repetition of a quiver.

Definition 6.3. *Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver with the sets Δ_0 and Δ_1 of vertices and arrows, respectively. The repetition of Δ , denoted by $\mathbb{Z}\Delta$, is a quiver which is defined as follows:*

- $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$
- $(\mathbb{Z}\Delta)_1 = \mathbb{Z} \times \Delta_1 \cup \sigma(\mathbb{Z} \times \Delta_1)$ with arrows $(n, \alpha) : (n, x) \rightarrow (n, y)$ and $\sigma(n, \alpha) : (n-1, y) \rightarrow (n, x)$ for each arrow $\alpha : x \rightarrow y$ in Δ_1 and $n \in \mathbb{Z}$.

Theorem 6.4. [25, Theorem 4.12] *Let Λ be a CM-finite algebra and let Γ be a component of $\Gamma_{\mathcal{S}}^s$ containing a boundary vertex. Then the following statements hold.*

- (1) *If the cardinality of Γ is finite, then $\Gamma = \mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver and G is an automorphism group of $\mathbb{Z}\Delta$ containing a positive power of the translation.*
- (2) *If Γ is infinite, then Γ is a stable tube.*

In particular, if $\mathcal{S}_G(\Lambda)$ is of finite representation type, then $\Gamma_{\mathcal{S}}^s$ is a disjoint union of the finite components containing a boundary vertex.

To state our next result we need some notations. Let G be an indecomposable non-projective Gorenstein projective Λ -module. The unique component of the stable Auslander-Reiten quiver $\Gamma_{\mathcal{S}}^s$ containing the vertex $(0 \rightarrow G)$ will be denoted by $\Gamma_{\mathcal{S}}^s(G)$.

Notation 6.5. *Let U be the complete set of all pairwise non-isomorphic indecomposable non-projective Gorenstein projective modules. Set*

$$\mathcal{O} := \{[G]_{\vartheta} \mid G \in U\},$$

where $\vartheta = \tau_G \Omega_{\Lambda}^{-1} \tau_G^2$ to be the set of all ϑ -orbits of the stable category $\underline{\text{Gprj}}\text{-}\Lambda$ and

$$\mathcal{C} := \{\Gamma_{\mathcal{S}}^s(G) \mid G \in U\},$$

to be the set of all components of the stable Auslander-Reiten quiver $\Gamma_{\mathcal{S}}^s$ containing a vertex of the form $(0 \rightarrow G)$, where $G \in U$.

There exists a well-defined map $\delta : \mathcal{O} \rightarrow \mathcal{C}$ sending $[G]_{\vartheta}$ to $\Gamma_{\mathcal{S}}^s(G)$.

Let \mathcal{C}^{∞} denote the subset of \mathcal{C} consisting of all infinite components and \mathcal{O}^{∞} denote the inverse image of \mathcal{C}^{∞} under the map δ .

Proposition 6.6. *With the above notation, the following statements hold.*

- (1) *The map δ is surjective.*
- (2) *The restricted map $\delta' : \mathcal{O}^{\infty} \rightarrow \mathcal{C}^{\infty}$ is bijective.*

6.2 Three-periodicity phenomenon

In the following, we provide some information concerning the cardinality of the finite components of the stable Auslander-Reiten quiver $\Gamma_{\mathcal{S}}^s$ containing a boundary vertex, where another kind of 3-periodicity phenomenon occurs.

Theorem 6.7. *[25, Theorem 6.1] Let Λ be a finite dimensional algebra over an algebraically closed field and let $\mathcal{S}_{\mathcal{G}}(\Lambda)$ be of finite representation type. Let Δ be a Dynkin diagram different from \mathbb{D}_4 and let G be a non-trivial group of weakly admissible automorphisms of $\mathbb{Z}\Delta$. If $\Gamma = \mathbb{Z}\Delta/G$ is a component of the stable Auslander-Reiten quiver $\Gamma_{\mathcal{S}}^s$, then 3 is a divisor of the cardinality of Γ .*

Note that the above proposition does not hold for type \mathbb{D}_4 . For $\Lambda = k[x]/(x^3)$, the stable Auslander-Reiten quiver $\Gamma_{\mathcal{S}}^s$ has only one component of type \mathbb{D}_4 with the cardinality 8, see e.g. [36, Section 6].

7 G -semisimple algebras

Let $\text{mod}-(\underline{\text{Gprj}}-\Lambda)$ be the category of finitely presented functors over the stable category $\underline{\text{Gprj}}-\Lambda$ of finitely generated Gorenstein projective Λ -modules. In [18], G -semisimple algebras are defined as those algebras Λ for which $\text{mod}-(\underline{\text{Gprj}}-\Lambda)$ is a semisimple abelian category. We studied some basic properties of such algebras. In particular, it is observed that the class of G -semisimple algebras contains important types of algebras, including gentle algebras and, more generally, quadratic monomial algebras. In [18], we also studied the CM-finiteness of path algebras over G -semisimple algebras. Finally, we extend Theorem 6.7 to the category of n -chains of monomorphisms.

7.1 Path algebras of G -semisimple algebras

Let $\mathcal{Q} = (V, E)$ be a quiver, i.e., a directed graph with the set of vertices V and the set of arrows E enhanced with a couple of functions s and t which assign, respectively, to any arrow a of its origin and terminal vertices. Let \mathcal{C} be an additive category. A representation \mathcal{X} of \mathcal{Q} by \mathcal{C} is obtained by associating to any vertex v an object \mathcal{X}_v in \mathcal{C} and to any arrow $a : v_1 \rightarrow v_2$ a morphism $\mathcal{X}_a : \mathcal{X}_{v_1} \rightarrow \mathcal{X}_{v_2}$ in \mathcal{C} . Such a representation is denoted as

$\mathcal{X} = (\mathcal{X}_v, \mathcal{X}_a)_{v \in V, a \in E}$, or simply $\mathcal{X} = (\mathcal{X}_v, \mathcal{X}_a)$. If \mathcal{X} and \mathcal{Y} are two representations of \mathcal{Q} , then a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is determined by a family $f = (f_v)_{v \in V}$ of \mathcal{C} in such a way that for any arrow $a : v_1 \rightarrow v_2$, the commutativity condition $f_{v_2} \mathcal{X}_a = \mathcal{Y}_a f_{v_1}$ holds. The representations of \mathcal{Q} by \mathcal{C} and the morphisms among them form a category that is denoted by $\text{rep}(\mathcal{Q}, \mathcal{C})$. In the case $\mathcal{C} = \text{mod-}\Lambda$, then $\text{rep}(\mathcal{Q}, \text{mod-}\Lambda)$ is just denoted by $\text{rep}(\mathcal{Q}, \Lambda)$.

Suppose that $\text{Gprj}(\mathcal{Q}, \Lambda)$ is the subcategory of $\text{rep}(\mathcal{Q}, \Lambda)$ consisting of all Gorenstein projective representations. The following result gives a local characterization of Gorenstein projective representations of an acyclic quiver.

Theorem 7.1. ([10, Theorem 3.5.1] or [32, Theorem 5.1]) *Let \mathcal{Q} be an acyclic quiver and \mathcal{X} a representation in $\text{rep}(\mathcal{Q}, \Lambda)$. Then \mathcal{X} is in $\text{Gprj}(\mathcal{Q}, \Lambda)$ if and only if*

- (i) *For each vertex v , \mathcal{X}_v is a Gorenstein projective Λ -module;*
- (ii) *For each vertex v , the Λ -morphism $\mathcal{X}^v : \bigoplus_{t(a)=v} \mathcal{X}_{s(a)} \rightarrow \mathcal{X}_v$ is a monomorphism whose cokernel is Gorenstein projective.*

Assume that $\pi : \text{Gprj-}\Lambda \rightarrow \underline{\text{Gprj-}}\Lambda$ is the canonical functor, where $\underline{\text{Gprj-}}\Lambda$ is the stable category of Gorenstein projective Λ -modules. For simplicity, for any morphism f in $\text{Gprj-}\Lambda$, let $\underline{f} := \pi(f)$, and to stress that we consider a Gorenstein projective module G in the stable category $\underline{\text{Gprj-}}\Lambda$, we denote it by \underline{G} . We define the following functor

$$\Psi := \Psi^{\mathcal{Q}} : \text{Gprj}(\mathcal{Q}, \Lambda) \rightarrow \text{rep}(\mathcal{Q}, \underline{\text{Gprj-}}\Lambda)$$

$$\mathcal{X} = (\mathcal{X}_v)_{v \in V} \mapsto \Psi(\mathcal{X}) := (\underline{\mathcal{X}}_v)_{v \in V}$$

$$f = (f_v)_{v \in V} : \mathcal{X} \rightarrow \mathcal{Y} \mapsto \Psi(f) = (\underline{f}_v)_{v \in V}$$

One should note that since each vertex of a projective presentation is a projective Λ -module, the functor Ψ induces the functor

$$\underline{\Psi} : \underline{\text{Gprj}}(\mathcal{Q}, \Lambda) \rightarrow \text{rep}(\mathcal{Q}, \underline{\text{Gprj-}}\Lambda),$$

where $\underline{\text{Gprj}}(\mathcal{Q}, \Lambda)$ is the stable category of $\text{Gprj}(\mathcal{Q}, \Lambda)$ modulo projective representations. The aim of this section is to prove that $\underline{\Psi}$ is an epivalence, extending the above epivalence within the framework of exact categories. Precisely, we prove the following result.

Theorem 7.2. [18, Theorem 4.2] *Let \mathcal{Q} be an acyclic quiver. Then $\underline{\Psi}$ is full and reflects isomorphisms. Moreover, $\underline{\Psi}$ is dense, whenever Λ is a G -semisimple finite dimensional algebra over an algebraic closed field k .*

Recall that a functor is said to be epivalence (or representation equivalence in the sense of Auslander) if it is full and dense which reflects isomorphisms. Therefore, the above theorem says that $\underline{\Psi}$ is an epivalence over G -semisimple algebras.

As an application of Theorem 7.2, we obtain the following Gabriel-style classification of Gorenstein projective representations over G -semisimple algebras.

Corollary 7.3. *Let \mathcal{Q} be an acyclic quiver, and let Λ be a G -semisimple finite-dimensional algebra over an algebraically closed field k . Let m be the number of isomorphism classes of indecomposable non-projective Gorenstein projective modules. Then the following statements hold.*

- (i) *The path algebra $\Lambda\mathcal{Q}$ is CM-finite if and only if \mathcal{Q} is Dynkin.*
- (ii) *The number of isomorphism classes of indecomposable non-projective Gorenstein projective representation is $m|\Phi^+|$, where $|\Phi^+|$ is the set of positive roots of the (corresponding) Dynkin diagram.*

7.2 The stable Auslander-Reiten quiver of path algebras over G -semisimple algebras

We denote by $\text{Gprj}(A_n, \Lambda)$ the category of Gorenstein representations of the linear quiver A_n . It is just $\text{Gprj}(A_n, \Lambda)$. In [18, Proposition 5.3], they provide a complete characterization of indecomposable Gorenstein projective representations in $\text{Gprj}(A_n, \Lambda)$ of the linear quiver A_n over a G -semisimple algebra Λ .

In the next theorem, We obtain insights into the size of the components of the stable Auslander-Reiten quiver $\text{Gprj}(A_n, \Lambda)$, which is a generalization of Theorem 6.7. Indeed, in Theorem 6.7, an interesting three-periodicity phenomenon was investigated. In what follows, we consider a generalization of this periodicity phenomenon for the finite components of the stable Auslander–Reiten quiver of $\mathcal{S}_n(\text{Gprj-}\Lambda)$, where Λ is a G -semisimple algebra.

Theorem 7.4. [18, Theorem 5.17] *Let Λ be a finite dimensional G -semisimple algebra over an algebraically closed field. Let Γ be a component of the stable Auslander-Reiten quiver of $\mathcal{S}_n(\text{Gprj-}\Lambda)$. Then*

- (1) *if n is even, then the size of Γ is divisible by $n + 1$;*
- (2) *if n is odd, then the size of Γ is divisible by $\frac{n+1}{2}$.*

8 Covering theory for monomorphism category

In a joint project with Hideto Asashiba and Razieh Vahed, we have been studying the covering theory for the monomorphism category. A preliminary version of this work is available on ArXiv, see [21].

Let \mathcal{A} be a locally bounded k -category and let G be a torsion-free group of k -linear automorphisms of \mathcal{A} acting freely on the objects of \mathcal{A} . Suppose $F : \mathcal{A} \rightarrow \mathcal{B}$ is a Galois functor. We naturally extend the push-down functor F_λ to a functor

$$\mathcal{S}F_\lambda : \mathcal{S}(\text{mod-}\mathcal{A}) \rightarrow \mathcal{S}(\text{mod-}\mathcal{B}),$$

between the corresponding monomorphism categories of $\text{mod-}\mathcal{A}$ and $\text{mod-}\mathcal{B}$.

For our purpose, we also need to extend the push-down functor to the stable categories, as visualized in the following diagram:

$$\begin{array}{ccc}
 \text{mod-}\mathcal{A} f_{F_\lambda}^{F_\lambda \circ h} \delta_h^F & & \text{mod-}\mathcal{B} \\
 \downarrow \pi & & \downarrow \pi \\
 \underline{\text{mod-}}\mathcal{A} & \begin{array}{c} \xrightarrow{F_\lambda \circ h} \\ \Downarrow \delta_h^F \\ \xrightarrow{F_\lambda} \end{array} & \underline{\text{mod-}}\mathcal{B}.
 \end{array}$$

Here π denotes the corresponding quotient functors.

Proposition 8.1. [21, Proposition 2.6] *Keep the notation from the above diagram. Then the following hold:*

- (i) *The functor $F_\lambda : \underline{\text{mod-}}\mathcal{A} \rightarrow \underline{\text{mod-}}\mathcal{B}$ is a G -precovering.*
- (ii) *If \mathcal{A} is locally support-finite, then F_λ is a G -Galois covering.*

We also need to consider how the push-down functor SF_λ , which we proved to be a G -precovering, induces a G -precovering for the functor categories as in the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{S}(\text{mod-}\mathcal{A}) & \xrightarrow{\Psi_{\text{mod-}\mathcal{A}}} & \text{mod-}(\underline{\text{mod-}}\mathcal{A}) \\
 \downarrow SF_\lambda & & \downarrow \Phi \\
 \mathcal{S}(\text{mod-}\mathcal{B}) & \xrightarrow{\Psi_{\text{mod-}\mathcal{B}}} & \text{mod-}(\underline{\text{mod-}}\mathcal{B}).
 \end{array}$$

The horizontal functors are similar to those defined in Construction 2.1. Based on this diagram, we are able to transfer the covering theory from the setting of module categories, which is well-understood, to the exact categories of monomorphism categories.

Later, in an independent work [16], we provide a systematic way to induce G -precoverings by adjoint functors using a 2-categorical machinery, which unifies many similar constructions of G -precoverings. In this project, we plan to apply our formulation of a covering theory for the monomorphism category to characterize the representation type of monomorphism categories over a specific class of algebras. This approach has already been applied in several papers in the literature, especially in the case $\Lambda_n = k[x]/(x^n)$.

By using covering techniques together with Auslander–Reiten theory, the authors of [36] showed that $\mathcal{S}(\Lambda_n)$ is of finite representation type when $n \leq 5$, and of wild representation type when $n \geq 7$. For $n = 6$, they proved that $\mathcal{S}(\Lambda_6)$ is of tame representation type and classified all indecomposable objects in it.

Suppose k is an algebraically closed field of characteristic different from 2. As an application of our works on covering theory over monomorphism categories, we want to determine

for which types of representation-finite selfinjective algebras the monomorphism category is of finite representation type (or equivalently, when the stable Auslander algebra is of finite representation type).

Let Λ be a hereditary finite-dimensional k -algebra and let M be a tilting Λ -module. Then $B = \text{End}_\Lambda(M)$ is called a *tilted algebra*. If $\Lambda \simeq k\Delta$, where $k\Delta$ is the path algebra of a quiver Δ over k , we say that B is a tilted algebra of type Δ . In particular, if the underlying graph of Δ is a Dynkin diagram, then B is called a tilted algebra of Dynkin type, which is representation-finite [26, Chapter III, Section 5].

In a preliminary version of this project, the following theorem was proved. However, in our joint reconsideration with Hideto Asashiba, we are planning to check this theorem more carefully. The work is still in progress, and we also plan to use the covering theory to study monomorphism categories over certain specific algebras, such as the Nakayama self-injective algebra with two simple modules.

Theorem 8.2. [21, Theorem 7.8] *Let Λ be a tilted algebra of Dynkin type Δ , and let ϕ be an admissible auto-equivalence of the repetitive category $\widehat{\Lambda}$. Assume A , resp. Γ , is the stable Auslander algebra of the path algebra $k\Delta$, resp. the representation-finite selfinjective algebra $\widehat{\Lambda}/\phi$. Then the following statements are equivalent:*

- (1) *The monomorphism category $\mathcal{S}(\text{mod-}\widehat{\Lambda}/\phi)$ is of finite representation type.*
- (2) *The monomorphism category $\mathcal{S}(\text{mod-}\widehat{\Lambda})$ is of finite representation type.*
- (3) *The stable Auslander algebra Γ is representation-finite.*
- (4) *The bounded derived category $\mathbb{D}^b(\text{mod-}A)$ is locally bounded.*
- (5) *The quiver Δ is of Dynkin type \mathbb{A}_n with $n \leq 4$.*

9 On the monomorphism category of large modules

Let R be an associative ring with identity. In [17], the authors investigated the structure of the monomorphism category of large R -modules and establish relation with the category of contravariant functors defined on finitely presented R -modules. Building upon the functors introduced by Ringel and Zhang [35], they constructed a functor from the monomorphism category of pure-injective modules, denoted by $\mathcal{S}(R\text{-Pinj})$, to the category of all (not necessarily finitely presented) contravariant functors on the category of finitely presented R -modules, namely $((R\text{-mod})^{\text{op}}, \mathcal{A}b)$.

The results of this paper emphasize the role of pure-injective modules in the study of homological properties of functor categories. In this context, we make use of the structure of flat modules and the notion of special flat resolutions in $((R\text{-mod})^{\text{op}}, \mathcal{A}b)$, as developed by Crawley-Boevey [4] and Herzog [27]. In particular, Herzog showed that every object in this category admits a special flat resolution of length at most two. This is very essential

to define a functor

$$\Psi : \mathcal{S}(R\text{-Mod}) \rightarrow ((R\text{-mod})^{\text{op}}, \mathcal{A}b)$$

This functor is indeed a big version of the functor defined in subsection 2.1.

Theorem 9.1. [17, Theorem 4.2 and Corollary 4.3] *The restriction of the functor Ψ to $\mathcal{S}(R\text{-Pinj})$, i.e.*

$$\Psi| : \mathcal{S}(R\text{-Pinj}) \rightarrow ((R\text{-mod})^{\text{op}}, \mathcal{A}b),$$

is full, dense and objective. In particular, the functor $\Psi|$ induces an equivalence Ψ' of categories

$$\Psi' : \mathcal{S}(R\text{-Pinj})/\mathcal{V} \xrightarrow{\sim} ((R\text{-mod})^{\text{op}}, \mathcal{A}b),$$

where \mathcal{V} denotes the smallest additive subcategory of $\mathcal{S}(R\text{-Pinj})$ containing all objects of the form $(E \xrightarrow{1} E)$ and $(0 \rightarrow E)$, where E is a pure-injective module.

A dual version of the above theorem is also considered in [17]. For this, let

$$\begin{aligned} \mathcal{F}(R\text{-Mod}) &= \text{All epimorphisms in } \mathbf{H}(R\text{-Mod}) \\ \mathcal{F}(R\text{-Pinj}) &= \text{the subcategory of } \mathcal{F}(R\text{-Mod}) \text{ consisting of all epimorphisms} \\ &\quad (Y \xrightarrow{g} Z) \text{ such that } \text{Kerg} \text{ and } Y \text{ are pure-injectives.} \end{aligned}$$

We also need to define a functor $\Phi : \mathcal{F}(R\text{-Pinj}) \rightarrow (\text{mod-}R, \mathcal{A}b)$ by using tensor product. Then, we have the following theorem.

Theorem 9.2. [17, Theorem 4.4 and Corollary 4.5] *The functor $\Phi : \mathcal{F}(R\text{-Pinj}) \rightarrow (\text{mod-}R, \mathcal{A}b)$ is full, dense and objective. In particular, the functor Φ induces an equivalence of categories*

$$\Phi' : \mathcal{F}(R\text{-Pinj})/\mathcal{U} \xrightarrow{\sim} (\text{mod-}R, \mathcal{A}b),$$

where \mathcal{U} denotes the smallest additive subcategory of $\mathcal{F}(R\text{-Pinj})$ consisting of all objects of the form $(E \xrightarrow{1} E)$ and $(E \rightarrow 0)$.

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