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A comprehensive study on chain conditions in Rings and Modules

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ABSTRACT

This survey presents a comprehensive and systematic overview of the theory of chain conditions on modules and rings with particular emphasis on injective and quasi-injective modules, as well as associated structural properties of rings. We cover classical and modern perspectives on ascending and descending chain conditions (acc and dcc) on submodules, essential submodules, and quotient modules. Key classes such as Noetherian, Artinian, seminoetherian, and isonoetherian modules and rings are explored in detail. We review the theory of quasi-injective modules under chain conditions, including characterization theorems, endomorphism ring decompositions, and their connections to quasi-Frobenius rings. The notions of essential Noetherian and essential Artinian modules and their significance in generalizing classical results. Finally, we touch upon divisibility properties in submodule chains, power series ring extensions, projective modules over semilocal rings, and related topics, highlighting open problems and future research directions.

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1   Classical History

Module theory originated in the early 20th century as a generalization of vector spaces and abelian groups to structures over rings rather than fields. The concept was introduced by Emmy Noether in the 1920s, who extended methods from commutative algebra and ring theory to study modules, creating a framework to analyze algebraic structures through submodules, quotient modules, and homomorphisms.

The development of module theory paralleled advances in ring theory and homological algebra, with key milestones including the classification of finitely generated modules over principal ideal domains and the use of exact sequences and category theory to understand module behavior. Over time, module theory became foundational in many areas of mathematics, including algebraic geometry, representation theory, and algebraic topology, linking algebraic structures with geometric and topological concepts. In the 19th century, the origins of chain conditions can be traced back to the ideal theory of rings, particularly in the study of factorization. Richard Dedekind, studied ideals in rings of algebraic integers and introduced the concept of Dedekind domains. The concept of chain conditions originated from efforts to understand finiteness in algebraic systems, notably through Emmy Noether's pioneering work in the 1920s and 1930s. She introduced the Noetherian condition by defining rings in which every ascending chain of ideals stabilizes, thereby forbidding infinite strictly increasing sequences of ideals. This condition made possible the rigorous classification and structural analysis of rings and their modules. Emil Artin complemented this with the study of Artinian rings, defined by the ascending stabilization of descending chains of ideals. This notion helped extend the understanding of rings and modules satisfying the descending chain condition (dcc). In mid 20th century, once ring theory matured, module theory naturally developed in parallel. Algebraists extended the concepts of Noetherian and Artinian rings to Noetherian and Artinian modules. A module is called Noetherian if it satisfies the ascending chain condition on submodules and a module is called Artinian if it satisfies the descending chain condition on submodules. This allowed the classification of modules based on their structural behavior and provided criteria for finiteness, simplicity, and decomposition. Modules of finite length are those modules that have a finite composition series, which is a finite chain of submodules each factor of which is simple. Such modules satisfy both the ascending chain condition (Noetherian) and descending chain condition (Artinian) on submodules, making them finitely generated and having a well-defined length equal to the number of simple factors in any composition series. Modules of finite length are fundamental in module theory and closely connected to the Jordan–Hölder theorem, which ensures the uniqueness of composition factors up to isomorphism and ordering. Additionally, finite length modules behave well with respect to exact sequences, direct sums, and submodules, making them important in structural and homological algebra contexts. The Jordan–Hölder Theorem, first developed in group theory in the late 19th and early 20th centuries, says that any two composition series of a finite group have the same number and types of simple factors (up to order and isomorphism). This idea was later applied to modules and other algebraic structures, becoming a useful tool for understanding and classifying them based on their simple building blocks. The Krull–Schmidt Theorem, developed in the early to mid-20th

century by Wolfgang Krull and Otto Schmidt, says that a module (or similar object) can be broken down into a finite sum of smaller pieces called indecomposable modules, and this breakdown is unique up to order and isomorphism. The Hilbert Basis Theorem, originally formulated for Noetherian rings by David Hilbert in 1890, was extended to Noetherian modules as algebraists broadened the finiteness concepts from ideals to submodules. This extension asserts that if a module is Noetherian (satisfies the ascending chain condition on submodules), then certain module constructions, like polynomial extensions, also remain Noetherian. The Hopkins–Levitzki Theorem states that for a semiprimary ring R, every left R-module that is either Artinian (satisfies the descending chain condition) or Noetherian (satisfies the ascending chain condition) has a composition series, making these module conditions equivalent under this ring condition. The theorem, proved independently by Charles Hopkins and Jacob Levitzki in 1939, generalizes earlier commutative results by Akizuki and establishes that Artinian rings are also Noetherian. This result is fundamental in ring and module theory, linking finiteness conditions and structural decomposition properties of modules over semiprimary rings.

2 Introduction

Chain conditions are fundamental tools in module and ring theory, providing structural control and enabling the classification of algebraic objects. The classical ascending chain condition (acc) and descending chain condition (dcc) on submodules underpin the theory of Noetherian and Artinian modules and rings, and have extensive applications in algebraic geometry, representation theory, and homological algebra.

This survey aims to assemble, clarify, and present recent developments and classical results concerning chain conditions in module theory, especially focusing on essential submodules, quasi-injective and injective modules, and specialized rings. We cover generalizations of Noetherian and Artinian conditions, such as seminoetherian and isonoetherian structures, explore the structure of big projective modules over noetherian semilocal rings, and investigate structural properties of endomorphism rings of quasi-injective modules with chain conditions. In [49], Zeyada et al. explore new characterizations of Noetherian rings and modules, emphasizing soc-injectivity and s-injectivity. They extend recent results on essential extensions of direct sums of injective modules, illustrating how these properties relate to Noetherian and quasi-Frobenius rings. The study further investigates conditions under which essential extensions of injective hulls of simple or singular modules remain direct sums of injective or projective modules, providing deeper insight into the structural properties of Noetherian rings and their modules. In [1], Ahmed et al. explores the relationship between descending chains of modules and the Jordan–Hölder theorem, aiming to provide a unified framework that connects concepts such as unique factorization domains, the Krull–Schmidt theorem, and the Jordan–Hölder theorem through the study of congruences on module monoids, exact sequences, and conditions ensuring the existence and uniqueness of composition series in general module-theoretic contexts. In [34], Matlis extends the study of module theory over commutative Noetherian rings beyond finitely generated modules by introducing and analyzing modules that satisfy the descending chain condition (dcc). It develops the foundational theory of such modules, explores their struc-

ture through functors and injective envelopes, and establishes dual relationships between Artinian modules and finitely generated modules using the theory of injective modules and the Koszul complex. The paper also introduces concepts such as consequences, dimension, and maximal orders, providing dual analogues to classical notions like sequences and codimension, and demonstrates how these dualities yield a unified understanding of the homological and structural properties of modules over Noetherian rings. For more information and fundamental results on chain condition on modules, readers are referred to [4], [29], [1], [15], [42], [41], [43], [26], [38], [2], and [3].

We will refer [31], [47], [48] and [30] for all the basic terminologies and notations. In this paper, we consistently denote $J(R)$, $Soc(M)$, $Z(_RR)$, $E(R)$, $T(M)$, $Tr(M)$, Spec(M), Ass(M) and Supp(M) as Jacobson radical of a ring, socle of a module, singular ideal of a ring, injective envelope of a ring, torsion submodule, trace of the module M, Spectrum of a module M, set of associated primes of M and support of module M respectively.

3   Chain condition on Modules and Rings

Throughout the paper, let R be a ring with identity, and all modules are assumed to be unital left R-modules unless otherwise specified.

Definition 3.1. *[35], A module M satisfies the ascending chain condition (acc) on submodules if every increasing chain*

$$N_1 \subseteq N_2 \subseteq \cdots$$

eventually stabilizes; that is, there exists k such that $N_k = N_{k+1} = \cdots$. Dually, M satisfies the descending chain condition (dcc) *on submodules if every decreasing chain of submodules becomes constant after finitely many steps.*

Definition 3.2. *[35], A module M is called Noetherian if it satisfies acc on submodules, and Artinian if it satisfies dcc on submodules. For example, finite-dimensional vector spaces K^n over a field K, simple modules, and semisimple modules are all both Artinian and Noetherian. It is important to observe that, in general, Noetherian modules are not necessarily Artinian, and Artinian modules are not necessarily Noetherian. For example, $\mathbb{Z}$ as a $\mathbb{Z}$-module is Noetherian but not Artinian, whereas the Prüfer p-group $\mathbb{Z}_{p^\infty}$ is Artinian but not Noetherian.*
These survey on chain condition in module theory studied the fundamental concepts of Noetherian and Artinian modules and rings. We explore the ascending chain condition (acc) and descending chain condition (dcc) for submodules and ideals, and show how these properties are central to understanding the structure of modules and rings. The review also discusses composition series, the Hilbert Basis Theorem, and provides several key examples and applications, such as the behavior of submodules, quotient modules, and polynomial rings over Noetherian rings.

In [20], Faith builds directly on the structure theorem that any simple Artinian ring Q is isomorphic to D_n, a full matrix ring over a division ring D. His goal was to characterize subrings $R \subseteq Q$ whose classical left quotient ring is Q itself, defining such R as left orders.

Two left orders R and S are said to be *equivalent* if there exist regular elements $a, b, a', b' \in Q$ satisfying

$$aRb \subseteq S \quad \text{and} \quad a'Sb' \subseteq R.$$

Faith proved that this relation is reflexive, symmetric, and transitive, allowing left orders to be grouped into equivalence classes independent of the ambient ring. A ring R is called simple if it has no nonzero proper two-sided ideals. A ring R is called simple Artinian if it is simple and satisfies the descending chain condition on ideals. For example, the matrix ring $M_n(D)$ over a division ring D is a simple Artinian ring. Below, we present several important results and properties that were examined in the preceding paper.

Theorem 3.3. *[20, Theorem 1], if R is a ring order in the simple Artinian ring $Q = D_n$, Then:*

1. *$U = B \cap P = A \cap P$ is an ideal of $P = R \cap D$;*

2. *$B \cap A \supseteq U_n \supseteq BA \supseteq U_n^2$;*

3. *P, U, U^2 are left order in D;*

4. *R, U_n^2, U_n, P_n are equivalent left order in D;*

5. *if $0 \neq u \in U$, then $R(U^{-1}) = P_n'$ where $P' = P(U^{-1})$.*

Theorem 3.4. *[20, Theorem 2], If P in Theorem 3.3 is a simple ring with identity, then:*

1. *$B, A, BA = T$, and U^2 are all simple rings.*

2. *$T = U_n^2$.*

3. *If $0 \neq u \in U^2$, then $P' = P[u^{-1}]$ and P_n' are simple rings.*

In [21], Faith starts by explaining some important ideas about annihilators and injective modules. For a module M over a ring R and a subset X of M, the right annihilator of X is the set of elements in R that make every element of X zero when multiplied by the right:

$$X_R = \{r \in R \mid Xr = 0\}.$$

Similarly, the left annihilator of a subset X of R is the set of elements in M that become zero when multiplied by any element of X on the left:

$$_RX = \{m \in M \mid mX = 0\}.$$

All right and left annihilators form lattices, meaning they are closed under intersections. A *left annulet* is a left ideal that is also an annihilator. The *ascending chain condition (acc)* on left annulets means that any increasing sequence of these ideals eventually stops growing. A module M over a ring R is said to be injective if every R-homomorphism from a submodule of any R-module into M can be extended to the whole module. An R-module M is said to be self-injective if it is injective as an R-module over itself. For

example, every divisible group is self-injective as a $\mathbb{Z}$-module. A module M is Σ-injective if any infinite direct sum of copies of M is injective. This idea connects injectivity to the behavior of annihilators. Self-injective rings satisfy the ascending chain condition on annihilators. R is *left self-injective* if every homomorphism from a left ideal of R to R can be extended to a homomorphism from R to R. A ring R is called a *quasi-Frobenius (QF) ring* if it satisfies any of the following equivalent conditions:

1. R is left and right Artinian and self-injective.

2. Every projective R-module is injective, and every injective R-module is projective.

3. R is left and right Noetherian and self-injective.

For example, all semisimple Artinian rings are *quasi-Frobenius (QF) rings.* The following is a collection of significant properties and results referenced in the above paper.

Theorem 3.5. *[21, Theorem 1], If R is left Artinian (or Noetherian) and is also left self-injective, then R is a quasi-Frobenius (QF) ring.*

Recall from [21], an R-module M is said to be **projective** if every R-homomorphism from M to a quotient module can be lifted to the corresponding module. For example, every free module is projective i.e., $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. We denote by $\mathcal{F}$ the category of all left R-module.

Theorem 3.6. *[21, Theorem 5], The following conditions on a ring R are equivalent:*

1. Every countably generated projective module in $\mathcal{F}$ is injective.

2. R is left self-injective and satisfies the ascending chain condition (acc) on left annulets.

3. Every projective module in $\mathcal{F}$ is injective.

Recall from [21], a ring R is called *semiprimary* if its Jacobson radical $\operatorname{rad} R$ is nilpotent and the quotient $R/\operatorname{rad} R$ is semisimple. A ring R is said to be *(left) perfect* if every left R-module has a projective cover. For example, every Artinian ring and every semisimple ring is both semiprimary and perfect.

Proposition 3.7. *[21, Proposition 1], If R is a perfect ring and satisfies the ascending chain condition (acc) on left annulets, then R is semiprimary.*

Theorem 3.8. *[21, Theorem 2], A ring R is quasi-Frobenius (QF) if and only if it is left self-injective and satisfies the ascending chain condition (acc) on left annulets.*

In [33], let R be a ring and A, B be ideals of R. The *colon ideal* of A by B is defined as

$$(A : B) = \{\, r \in R \mid rB \subseteq A \,\}.$$

It consists of all elements $r \in R$ such that multiplying r by every element of B results in an element of A. If $B = (x)$ is a principal ideal generated by an element $x \in R$, then

$$(A : x) = \{\, r \in R \mid rx \in A \,\}.$$

A module M is said to satisfy the *(accr)* condition (respectively, *(accr*)*) if every ascending chain of residuals of the form

$$(N : B) \subseteq (N : B^2) \subseteq (N : B^3) \subseteq \cdots$$

terminates for every submodule N of M and every finitely generated (respectively, principal) ideal B of R. Below, we present several essential properties and results drawn from the preceding paper.

Lemma 3.9. *[33, Lemma 1], Let M be an R-module, $r \in R$, and $n \in \mathbb{Z}^+$ (the set of positive integers). Then the following statements are equivalent:*

1. *The ascending chain of submodules $\{((0) : r^k)\}_{k \in \mathbb{Z}^+}$ terminates at n;*

2. *$((0) : r^n) = ((0) : r^{n+1})$;*

3. *$((0) : r^n) \cap r^n M = (0)$.*

Proposition 3.10. *[33, Proposition 1], Let N be a submodule of an R-module M, and let $r \in R$. Then the following statements are equivalent:*

1. *The ascending chain of submodules $\{(N : r^k)\}_{k \in \mathbb{Z}^+}$ terminates;*

2. *There exists an $n \in \mathbb{Z}^+$ such that $(N : r^n) = (N : r^{n+1})$;*

3. *There exists an $n \in \mathbb{Z}^+$ such that $N = (N : r^h) \cap (N + r^h M)$ for every $h > n$.*

Theorem 3.11. *[33, Theorem 1], For any R-module M, the following statements are equivalent:*

1. *M satisfies (accr);*

2. *M satisfies (accr*);*

3. *For any submodule N of M and any element $r \in R$, there exists a positive integer n such that $N = (N : r^h) \cap (N + r^h M)$ for every $h > n$;*

4. *For any submodule N of M and any finitely generated ideal B of R, there exists a positive integer n such that $N = (N : B^h) \cap (N + B^h M)$ for every $h > n$.*

Recall from [33], let N be a submodule of an R-module M, and let B be an ideal of R. We say that the *Artin-Rees property* holds for N and B if there exists a positive integer n such that

$$N \cap B^h M \subseteq BN \quad \text{for all } h > n.$$

A module M is called an *Artin-Rees module* if the Artin-Rees property holds for every submodule of M and every ideal of R. For example, $\mathbb{Z}^n$ as a $\mathbb{Z}$-module is *an Artin-Rees module.*

Remark: Every finitely generated module M over a Noetherian ring satisfies the Artin-Rees property.

Theorem 3.12. *[33, Theorem 2], If an R-module M satisfies the* (accr) *condition, then the Artin-Rees property holds for every submodule N of M and every finitely generated ideal B of R.*

Theorem 3.13. *[33, Theorem 3], Let M be an R-module that satisfies the* (accr) *condition. Then, for every finitely generated ideal B of R,*

$$\bigcap_{n=1}^{\infty} B^n M = \{0\}.$$

Consequently, if either B is a finitely generated ideal contained in the Jacobson radical of R, or R is an integral domain over which M is torsion-free, then

$$\bigcap_{n=1}^{\infty} B^n M = (0).$$

In [24], Jain et al. establish that a left continuous ring satisfying the ascending chain condition (acc) on essential left ideals is necessarily left Artinian. It builds upon previous results for self-injective rings by replacing self-injectivity with continuity, thereby generalizing the condition under which chain conditions on essential ideals ensure Artinianness. The authors also present examples demonstrating that one-sided continuity or chain conditions do not necessarily imply the same properties on the opposite side, highlighting the asymmetry in the behavior of such rings. For example, let R be a ring that has exactly three left ideals: 0, $J(R)$, and R, and assume that R is left artinian with composition length 3 (see [19, Example 7.11'.2, p. 338]). The ring R is clearly left continuous. However, R is not right continuous. Some important properties and results are mentioned below from the above paper.

Lemma 3.14. *[24, Lemma 3], For any left module M, the module M satisfies the* ascending chain condition (acc) *on* essential submodules *if and only if the* factor module $M/\operatorname{Soc}(M)$ *is* left Noetherian.

Theorem 3.15. *[24, Theorem], Let R be a ring that satisfies the* ascending chain condition (acc) *on* essential left ideals. *Then:*

1. *If R is* left *or* right continuous, *then R is* semiperfect.

2. *If R is* left continuous, *then R is* left Artinian.

3. *If R is* right continuous, *it does not necessarily follow that R is* left Artinian.

In [16], modules that satisfy the ascending chain condition (acc) or descending chain condition (dcc) on essential submodules, with particular emphasis on quasi-injective and quasi-projective modules. Below, we present several important results and properties that were examined in the preceding paper.

Lemma 3.16. *[16, Lemma 2], For a left R-module M, the following conditions are equivalent:*

1. *M satisfies the ascending chain condition on essential submodules;*

2. *The quotient $M/\operatorname{Soc}(M)$ is Noetherian.*

Proposition 3.17. *[16, Proposition 3], Let M be a finitely generated quasi-injective left R-module.*

1. *If M satisfies the ascending chain condition (acc) on essential submodules, then M is Noetherian.*

2. *If M satisfies the descending chain condition (dcc) on essential submodules, then M is Artinian.*

Example 3.18. *Let $M = \mathbb{Z}/p^n\mathbb{Z}$ for a prime p. This module is injective over itself and clearly satisfies both the ascending chain condition and descending chain condition on submodules, hence it is both Noetherian and Artinian.*

Example 3.19. *A vector space V over a field F is injective as an F-module. If V is finite-dimensional, then it satisfies the ascending chain condition and the descending chain condition on subspaces, making it both Noetherian and Artinian. For infinite-dimensional V, neither chain condition holds.*

In [16], Dung et al. also proved that the endomorphism ring of such modules decomposes into a direct sum of a left Artinian ring and a von Neumann regular, left self-injective ring linking module chain conditions to the structure of *quasi-Frobenius rings*.

Herbera and Prhoda [23] studied the structure of projective modules over Noetherian semilocal rings. They showed that the isomorphism classes of big projective modules form a *monoid* that can be described using solutions to linear Diophantine equations. For example, over a semilocal ring with two maximal ideals, the classes of projective modules can correspond to lattice points in $\mathbb{Z}^2$ subject to congruence conditions. This connection ties algebra to number theory. Several important results and properties from the above paper are presented below.

Proposition 3.20. *[23, Proposition 2.1], Let R be a Noetherian ring. Then, for any two-sided ideal I of R, the following conditions are equivalent:*

1. *$I^2 = I$;*

2. *There exists a countably generated projective left R-module P such that the trace ideal of P, denoted $\operatorname{Tr}(P)$, equals I;*

3. *For every finitely generated projective left R/I-module P', there exists a countably generated projective left R-module P satisfying $P/PI \cong P'$ and $I \subseteq \text{Tr}(P)$;*

4. *There exists a countably generated projective left R-module Q such that $\text{Tr}(Q) = I$;*

5. *For every finitely generated projective left R/I-module Q', there exists a countably generated projective left R-module Q such that $Q/IQ \cong Q'$ and $I \subseteq \text{Tr}(Q)$.*

Essential submodules and uniform modules are particularly important when studying chain conditions on more restricted families of submodules.

Definition 3.21. *[37], A submodule N of M is* essential *in M (written $N \subseteq_e M$) if $N \cap K \neq 0$ for every nonzero submodule K of M and an R-module M is called* uniform *if any two nonzero submodules of M have a nontrivial intersection. For example, every simple module is uniform.*

Here, essential submodules are those that intersect every nonzero submodule nontrivially. In [8], Chakraborty et al. introduced essential Noetherian and essential Artinian modules, which play a fundamental role in the modern generalizations of chain conditions. Essential Noetherian (Artinian) modules generalize the classical notions of Noetherian (Artinian) modules and provide a deeper understanding of how modules behave under decomposition, quotienting, and homomorphisms. Specifically, a module M over a ring R is called essentially Noetherian (Artinian) if every essential submodule of M satisfies the ascending (descending) chain condition on submodules. This survey presents some important theorems along with brief proof strategies.

Example 3.22. *Consider $\mathbb{Z}$ as a module over itself. Every nonzero submodule is of the form $n\mathbb{Z}$. All of these are essential in $\mathbb{Z}$ because $n\mathbb{Z} \cap m\mathbb{Z} = \text{lcm}(n, m)\mathbb{Z}$, so it trivially satisfies the ascending chain condition and the descending chain condition on essential submodules.*

Example 3.23. *In a uniform module (where every pair of nonzero submodules intersects nontrivially), the only essential submodule is the whole module itself. Thus, any uniform module automatically satisfies both ascending chain condition and the descending chain condition on essential submodules. Below, we list several important properties and results referenced in the preceding paper.*

Proposition 3.24. *[8, Proposition 1.2], A module M is e-Artinian if and only if its essential submodules and quotient modules are e-Artinian.*

Theorem 3.25. *[8, Theorem 1.4], Let A and B be any modules where $A \subseteq B$ and B is an e-Artinian module. If M is a B-injective module, then M is e-Artinian.*

Proposition 3.26. *[36, Proposition 1], Let A and B be modules with $A \subseteq B$. If B is an e-Noetherian module and M is a B-injective module, then M is also an e-Noetherian module.*

Theorem 3.27. *[36, Theorem 7], Let M be a simple R-module, and let N be a Noetherian module. Then $M \oplus N$ is e-Noetherian.*

Theorem 3.28. *[36, Theorem, 11], Let M be an R-module. Then M is locally e-Noetherian if and only if the direct sum of an arbitrary family of M-injective modules is M-injective.*

Theorem 3.29. *[36, Theorem 1], Let M be an essential Noetherian module and $N \subseteq M$ be essential. Then N is essential Noetherian.*

In [17], Facchini and Nazemian extended the classical chain conditions by relaxing the requirement of stabilization from *equality* to *isomorphism*. This led to three new classes of modules:

- **Iso-Artinian modules**: every descending chain of submodules eventually stabilizes up to isomorphism. For example, $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}$-module is an iso-Artinian module. We note that every Artinian modules is iso-Artinian, but every iso-Artinian is not Artinian. For instance, the $\mathbb{Z}$-module $\mathbb{Z}$ is iso-Artinian, although it is not Artinian.

- **Iso-Noetherian modules**: every ascending chain of submodules eventually stabilizes up to isomorphism. For example, every finite-dimensional vector space is an iso-noetherian module.

- **Iso-simple modules**: every nonzero submodule is isomorphic to the whole module. For example, $\mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}$-module is an iso-simple module.

A module M is said to be essential isoartinian, essential isonoetherian, or essential isosimple according to the following systematic conditions:

1. M is an essential isoartinian if every descending chain of essential submodules of M stabilizes up to isomorphism.

2. M is an essential isonoetherian if every ascending chain of essential submodules of M stabilizes up to isomorphism.

3. M is an essential isosimple if every nonzero essential submodule of M is isomorphic to M itself.

The following are some of the key properties and results highlighted in the preceding paper and essential for the development of the results.

Proposition 3.30. *[17, Proposition 2.3], Let M be a projective iso-simple R-module and let N be an iso-Artinian (resp. iso-Noetherian) module. Then the direct sum $N \oplus M$ is also an iso-Artinian (resp. iso-Noetherian) R-module.*

Corollary 3.31. *Let N be an essential iso-Artinian (resp. essential iso-Noetherian) module and let M be a projective iso-simple R-module. Then the direct sum $N \oplus M$ is also an essential iso-Artinian (resp. essential iso-Noetherian) R-module.*

Proof. The proof is the same as [17, Proposition 2.3]. □

Lemma 3.32. *[17, Lemma 2.8], Let M be a left module that is uniserial and isosimple. Then the lattice of submodules of M is order anti-isomorphic to the ordinal ω^γ, where γ is the Krull dimension K.dim(M) of M.*

Lemma 3.33. *[17, Lemma 2.10], Let M be a finitely generated isoartinian module. If $M \cong M \oplus N$ for some module N, then N is Noetherian.*

Corollary 3.34. *Let M be a finitely generated and essential isoartinian R-module. If $M \cong M \oplus N$ for some module N, then N is essential Noetherian.*

Proof. The proof follows the same as in [17, Lemma 33]. $\qquad\qquad\qquad\qquad$ $\square$

Proposition 3.35. *[17, Proposition 2.12], An isoartinian module cannot contain a chain of cyclic submodules that is order-isomorphic to the linearly ordered set $\omega^{op} \times \omega$.*

Theorem 3.36. *[17, Proposition 2.13], An isonoetherian module cannot contain a chain of cyclic submodules that is order-isomorphic to the ordinal ω^2.*

Proposition 3.37. *[17, Proposition 3.4], Let M be an isosimple R-module over an arbitrary ring R, and let $E := \mathrm{End}_R(M)$ denote its endomorphism ring. Consider $L(M)$, the lattice of all submodules of M; $L(_E E)$, the lattice of all left ideals of E; and $L_p(_E E)$, the partially ordered subset of $L(_E E)$ consisting of all principal right ideals of E. For any $f \in E$, assign to the principal right ideal $fE \in L_p(_E E)$ the submodule $\Phi(fE) := f(M)$ of M. Then*

$$\Phi : L_p(_E E) \longrightarrow L(M)$$

is a well-defined isomorphism of partially ordered sets.

Corollary 3.38. *[17, Corollary 3.6], For the endomorphism ring E of an isosimple module M, the following conditions are equivalent:*

1. *$L_p(_E E)$ is a sublattice of $L(_E E)$.*

2. *E is a left Bézout domain.*

3. *E is a PRID.*

4. *M is an intrinsically projective module.*

Proposition 3.39. *[17, Proposition 4.2], A left isoartinian ring R is semiprime if and only if the intersection of the annihilators of all isosimple left R-modules is zero.*

Corollary 3.40. *An essential isoartinian ring R is semiprime if and only if the intersection of the annihilators of all isosimple R-module is zero.*

In [17], the *right annihilator* of an ideal A of a ring R is defined as $r.\mathrm{ann}(A) = \{\, r \in R \mid ar = 0 \text{ for all } a \in A \,\}$, and the *left annihilator* is $l.\mathrm{ann}(A) = \{\, r \in R \mid ra = 0 \text{ for all } a \in A \,\}$. In general, these two need not be the same; however, if R is *semiprime* (i.e., it has no nonzero nilpotent ideals), then they coincide. Indeed, if $x \in r.\mathrm{ann}(A)$, then $Ax = 0$ and the ideal $RAxR$ satisfies $(RAxR)^2 = 0$, which forces $RAxR = 0$ in a semiprime ring; hence

$xA = 0$, so $x \in l.\text{ann}(A)$. By symmetry the reverse inclusion also holds, giving $r.\text{ann}(A) = l.\text{ann}(A)$. Thus, for semiprime rings we may write $\text{ann}(A) := r.\text{ann}(A) = l.\text{ann}(A)$, and any ideal of this form is called an *annihilator ideal*.

Theorem 3.41. *[17, Theorem 4.6], A semiprime left isoartinian ring R, the following conditions are equivalent:*

1. *R is left Noetherian.*

2. *R is left Goldie.*

3. *R_R has finite Goldie dimension.*

4. *If S is an isosimple left ideal of R and I_S denotes the ideal generated by the sum of all left ideals isomorphic to S, then there exists $x \in I_S$ such that $r.\text{ann}(x) \cap I_S = 0$.*

5. *If S is an isosimple left ideal of R and I_S is the ideal generated by the sum of all left ideals isomorphic to S, then I_S is a finite direct sum of isosimple left ideals.*

6. *R_R is a (direct) sum of isosimple left ideals.*

Lemma 3.42. *[17, Lemma 4.10], Let R be a commutative ring, then we have:*

1. *If R is isoartinian, then it satisfies the ascending or descending chain conditions on annihilators.*

2. *If R is self-injective, then R is isoartinian if and only if it is Artinian.*

A. Facchini et al. [18], introduced the concept of isoartinian and isonoetherian modules generalize classical chain conditions by considering stabilization up to isomorphism. A module is called isoartinian if every descending chain of submodules eventually consists of mutually isomorphic submodules. Similarly, a module is isonoetherian if every ascending chain of submodules stabilizes up to isomorphism. The concept of an isosimple module extends the classical notion of simplicity: a module is isosimple if every nonzero submodule is isomorphic to the module itself. Below, we present several important results and properties that are discussed in the preceding paper.

Proposition 3.43. *[18, Proposition 3.2], Every isonoetherian module has finite uniform dimension.*

Corollary 3.44. *Every essential isonoetherian module has finite uniform dimension.*

Proof. The proof is essentially identical to that of [18, Proposition 3.2]. $\qquad\square$

Lemma 3.45. *[18, Lemma 3.5], Let R be a commutative ring, then we have:*

1. *If R is isonoetherian, then it satisfies the descending chain condition (ascending chain condition) on annihilators.*

2. *If R is self injective, then R is isonoetherian if and only if R is noetherian.*

Corollary 3.46. *Let R be a commutative ring:*

1. *If R is isoartinian, then it satisfies the ascending chain condition (descending chain condition) on annihilators.*

2. *If R is self injective, then R is isoartinian if and only if R is artinian.*

Proof. The proof is essentially identical to that of [18, Lemma 3.5]. $\square$

In [18], the isoradical I-rad$(_R R)$ of a ring is the intersection of the annihilators of all isosimple R-modules. An ideal I of a ring R is left T-nilpotent if, for every sequence $a_1, a_2, \cdots$ of elements of I, there exists a positive integer n such that $a_1 a_2 \cdots a_n = 0$. Clearly, every T-nilpotent ideal is nil. Let $\mathcal{U}$ denote the class of isosimple left R-modules, and let Gen$(\mathcal{U})$ be the class of all left R-modules M for which there exists an indexed family $(U_\alpha)_{\alpha \in A}$ with each $U_\alpha \in \mathcal{U}$, together with an epimorphism

$$\bigoplus_{\alpha \in A} U_\alpha \twoheadrightarrow M.$$

For any left module M, set $Tr_M(\mathcal{U}) := \sum\{h(U)|h : U \longrightarrow M$ is a homomorphism for some $U \in \mathcal{U}\}$. Thus $M \in Gen(\mathcal{U})$ if and only $Tr_M(\mathcal{U}) = M$. It is convenient to denote $Tr_M(\mathcal{U})$ by $I - soc(M)$.

Proposition 3.47. *[18, Proposition 4.2], A isoartinian ring R is semiprime if and only if I-rad$(_R R) = 0$.*

Theorem 3.48. *[18, Theorem 4.6], Let R be a ring whose cyclic left modules are all isoartinian. Then:*

1. *I-rad$(_R R)$.*

2. *$P(R)$ is left T-nilpotent.*

3. *If $P(R)$ is left T-nilpotent, then it is nilpotent.*

Corollary 3.49. *Let R be a ring whose cyclic left modules are all essential isoartinian. Then:*

1. *I-rad$(_R R)$.*

2. *$P(R)$ is left T-nilpotent.*

3. *If $P(R)$ is left T-nilpotent, then it is nilpotent.*

Proof. The proof is essentially identical to that of [18, Theorem 4.6]. $\square$

Theorem 3.50. *[18, Theorem 4.13], The following conditions are equivalent for a ring R:*

1. $I - soc(R) = R$;

2. R is a sum of isosimple left ideals;

3. For any left R-module M, $I - soc(M) = M$;

4. R is a finite direct product of prime left Noetherian rings, each of which is a sum of isosimple left ideals.

In [7], A left R-module M is called *ei-Noetherian* (respectively, *ei-Artinian*) if every ascending (respectively, descending) chain of essential submodules

$$M_1 \subseteq_e M_2 \subseteq_e M_3 \subseteq_e \cdots \quad (\text{resp.}, M_1 \supseteq_e M_2 \supseteq_e M_3 \supseteq_e \cdots)$$

stabilizes up to isomorphism; that is, there exists an index n such that $M_i \cong M_n$ for all $i \geq n$. For example, every simple and semi-simple module is an ei-Noetherian (ei-Artinian) module. The following are some of the key properties and results discussed in the paper above.

Proposition 3.51. *[7, Proposition 3.3], An R-module M is* ei-Noetherian *(respectively,* ei-Artinian*) if and only if the following equivalent conditions hold:*

1. *For every nonempty set $\mathcal{F}$ of essential submodules of M, there exists $N \in \mathcal{F}$ such that for every submodule $K \subseteq N$ (respectively, $K \supseteq N$) with $K \in \mathcal{F}$, we have $N \cong K$.*

2. *For every nonempty chain $\mathcal{C}$ of essential submodules of M, there exists $N \in \mathcal{C}$ such that for every submodule $K \subseteq N$ (respectively, $K \supseteq N$) with $K \in \mathcal{C}$, we have $N \cong K$.*

Proposition 3.52. *[7, Proposition 3.8], Let M be an ei-Noetherian (respectively, ei-Artinian) R-module. If $\operatorname{Soc}(M)$ is essential in M, then $M/\operatorname{Soc}(M)$ is isonoetherian (respectively, isoartinian).*

Recall from [7], an R-module M is called *regular* if every finitely generated submodule of M is a direct summand of M. For example, every projective module over von Neumann regular rings and every semisimple module is a regular module. Following [46], a module M is said to be *quasi polysimple* if every nonzero submodule of M contains a uniform submodule. For example, $\mathbb{Z}/n\mathbb{Z}$ as a $\mathbb{Z}$-module is a quasi polysimple.

Proposition 3.53. *[7, Proposition 3.14], If M is a regular R-module, then M satisfies the iso descending chain condition on finitely generated non-summand submodules, and every iso-retractable submodule of M is a direct summand of M.*

Proposition 3.54. *[7, Proposition 3.15], Let M be a quasi polysimple R-module such that $\operatorname{Soc}(M) \subseteq_e M$ and $M/\operatorname{Soc}(M)$ is projective. Then M satisfies the iso ascending chain condition on finite-dimensional non-summand submodules if and only if M is isonoetherian or every uniform submodule of M is a direct summand of M.*

Corollary 3.55. *Let M be a quasi polysimple R-module such that $\operatorname{Soc}(M) \subseteq_e M$ and $M/\operatorname{Soc}(M)$ is projective. Then M satisfies the essential iso ascending chain condition on finite-dimensional non-summand submodules if and only if M is essential isonoetherian or every uniform submodule of M is a direct summand of M.*

Proof. The proof is essentially identical to that of [18, Proposition 3.15]. □

In [7], Chaturvedi et al. present new generalizations of classical chain conditions in module and ring theory, introducing concepts such as nei-Noetherian and nei-Artinian modules and rings. These notions extend the traditional Noetherian and Artinian properties by focusing on chains of non-essential submodules. They provide various characterizations and illustrative examples, highlighting how these new conditions differ from previously studied concepts like isonoetherian and isoartinian modules. Additionally, examines the interaction of these chain conditions with module structure, uniform dimension, and properties of semiprime and semihereditary rings.

Definition 3.56. *An R-module M is called* nei-Noetherian *(respectively,* nei-Artinian*) if every ascending (respectively, descending) chain of non-essential submodules*

$$M_1 \subseteq_{ne} M_2 \subseteq_{ne} M_3 \subseteq_{ne} \cdots \quad (resp.,\ M_1 \supseteq_{ne} M_2 \supseteq_{ne} M_3 \supseteq_{ne} \cdots)$$

stabilizes up to isomorphism; that is, there exists an index n such that $M_i \cong M_n$ for all $i \geq n$ (respectively, $i \geq n$ in the descending case). For example, uniform modules and iso-retractable modules are nei-Noetherian and nei-Artinian.

Remark: Every isoartinian (isonoetherian) module is nei-Artinian (nei-Noetherian). But the converse is not true. For example, $\mathbb{Q}$ as $\mathbb{Z}$-module is nei-Artinian (nei-Noetherian) but not isoartinian (isonoetherian).

Theorem 3.57. *[7, Theorem 2.5], For an R-module M, the following statements are equivalent:*

1. *M is nei-Artinian.*

2. *Every non-essential submodule of M is isoartinian.*

3. *Every proper closed submodule of M is isoartinian.*

By [7], let S be a commutative ring and M an S-module. The *trivial extension* of M by S, denoted $[S, M]$, is the ring consisting of all pairs (a, x) with $a \in S$ and $x \in M$, equipped with addition

$$(a, x) + (b, y) = (a + b, x + y)$$

and multiplication

$$(a, x)(b, y) = (ab, xb + ya).$$

The subset $[0, M] = \{(0, x) : x \in M\}$ is an ideal of $[S, M]$ with $[0, M]^2 = 0$, and for any submodule $N \subseteq M$, $[0, N]$ is also an ideal. An S-module M is called *divisible* if for every nonzero $c \in S$, $M = Mc$, that is, every element of M is divisible by c. For example, $\mathbb{Q}$ as a $\mathbb{Z}$-module is divisible.

Proposition 3.58. *[7, Proposition 2.9], Let S be a commutative domain, M is a nonzero divisible S-module, and $R = [S, M]$. Then R is a nei-Noetherian (respectively, nei-Artinian) ring if and only if M is a nei-Noetherian (respectively, nei-Artinian) S-module.*

Proposition 3.59. *[7, Proposition 2.10], Let S be a commutative domain and M is a nonzero torsion-free S-module. If $R = [S, M]$, then R is a nei-Noetherian (respectively, nei-Artinian) ring if and only if M is a nei-Noetherian (respectively, nei-Artinian) S-module.*

Recall that from [7], a module M is *monoform* if every nonzero homomorphism from a submodule $N \subseteq M$ to M is a monomorphism. Also, following [39], a nonzero R-module M is called *compressible* if for every nonzero submodule $N \subseteq M$, there exists a monomorphism $f : M \to N$. For example, if R is any domain then every left ideal of R is a compressible R-module.

Proposition 3.60. *[7, Proposition 2.12], If M is a nei-Noetherian R-module, then its uniform dimension satisfies u-dim$(M) < 1$.*

In [5], Aydogdu et al. explore the ascending and descending chain conditions (acc and dcc) on non-summand submodules of modules over a ring, with a focus on identifying when such modules are Noetherian, Artinian, or semisimple. By analyzing various classes of modules, including finitely generated, finite dimensional, and cyclic modules, the study establishes necessary and sufficient conditions under which a module satisfies chain conditions on its non-summand submodules. Let R be a ring. A *class χ* of R-modules is defined as a collection of R-modules that includes the zero module and is closed under isomorphisms. If a module belongs to χ, it is referred to as an *χ-module*. Similarly, an *χ-submodule* (respectively, *χ-summand*, *χ-non-summand*) denotes an χ-module that is also a submodule (respectively, summand, non-summand) of M. The following are some of the key properties and results discussed in the paper above.

Proposition 3.61. *[5, Proposition 2.1], An R-module M satisfies the ascending (respectively, descending) chain condition on χ-submodules if and only if it satisfies the same condition on both χ-summands and χ-non-summands.*

Lemma 3.62. *[5, Lemma 2.4], Let χ be a class of R-modules that is closed under extensions, and let N be an χ-submodule of an R-module M. If M satisfies the ascending (respectively, descending) chain condition on χ-non-summands, then the quotient module M/N also satisfies the ascending (respectively, descending) chain condition on χ-non-summands.*

Lemma 3.63. *[9, Lemma 6], Let M be an e-Artinian R-module that satisfies the descending chain condition on χ-non-summands, where x is a class of R-modules closed under extensions. If K is an R-submodule of M, then the quotient module M/K also satisfies the descending chain condition on χ-non-summands.*

Lemma 3.64. *[5, Lemma 2.6], Let χ be a class of modules closed under finite direct sums, and let M be a module satisfying the ascending (respectively, descending) chain condition on χ-non-summands. Suppose L and N are submodules of M with $L \cap N = 0$. Then either L satisfies the ascending (respectively, descending) chain condition on χ-submodules, or every χ-submodule of N is a direct summand of M, and consequently also a direct summand of N.*

Theorem 3.65. *[5, Theorem 2.9], Let χ be a class of R-modules that is closed under finite direct sums and direct summands. Then an R-module M satisfies the descending chain condition on χ-non-summands if and only if every χ-non-summand of M satisfies the descending chain condition on χ-submodules.*

Theorem 3.66. *[5, Theorem 3.2], A module M satisfies the ascending chain condition on non-summands if and only if M is either semisimple or Noetherian.*

Recall from [5], let X denote the class of finitely generated R-modules. A module M is called *regular* if every finitely generated submodule of M is a direct summand. Clearly, regular modules satisfy both the ascending and descending chain conditions on finitely generated non-summands.

Theorem 3.67. *[5, Theorem 3.7], The following statements are equivalent for a R-module M:*

1. *M satisfies the ascending chain condition on finitely generated non-summands;*

2. *For every finitely generated non-summand L of M, the quotient M/L is Noetherian;*

3. *For every non-finitely generated submodule N of M, every finitely generated submodule of N is a direct summand of M.*

A module M is called *semiartinian* if every non-zero homomorphic image of M has a non-zero socle [5].

Theorem 3.68. *[5, Theorem 3.18], If a module M satisfies the descending chain condition on finitely generated non-summands, then there exists a semiartinian submodule S of M such that the quotient M/S is regular.*

In [9], Chakraborty et al. studied the essential Artinian rings and modules generalizing classical Artinian rings and modules. An R-module M is said to be essential Artinian (e-Artinian) if it satisfies the descending chain condition on essential submodules of M. A ring R is said to be essential Artinian if $M = {}_R R$ is a left essential Artinian module.

Theorem 3.69. *[9, Theorem 2.7], A semi-prime e-Artinian ring R is a Goldie ring.*

Proposition 3.70. *[9, Proposition 3], Let R be an e-Artinian ring. Then R is semi-primary if and only if R has no infinite set of orthogonal idempotents.*

Lemma 3.71. *[9, Lemma 11], Let R be an e-Artinian ring. If R is regular then it is semisimple.*

Theorem 3.72. *[9, Theorem 2.9], If R is an e-Artinian ring, then R[x] is also e-Artinian.*

In [14], Dastanpour et al. introduce and study modules that satisfy ascending and descending chain conditions based on the existence of epimorphisms between successive submodules, generalizing the classical Noetherian and Artinian conditions and characterizing rings and modules possessing such "epi-chain" properties.

Definition 3.73. *[14], Let M be an R-module. The module M is said to satisfy the epi-ascending chain condition (epi-acc) on submodules if, for every ascending chain of submodules*

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots,$$

there exists a natural number k such that for all $i \geq k$, there exists an epimorphism

$$\varphi_i : M_{i+1} \twoheadrightarrow M_i.$$

Similarly, a ring R is said to satisfy the left epi-acc if the left R-module $_RR$ satisfies epi-acc on its submodules.

Definition 3.74. *An R-module M is said to satisfy the mono-ascending chain condition (mono-acc) on essential submodules if, for every ascending chain of essential submodules $M_1 \subseteq_e M_2 \subseteq_e \cdots$, there exists a natural number k such that for all $i \geq k$, there exists a monomorphism $\phi : M_{i+1} \longrightarrow M_i$.*

Example 3.75. *[14, Example 2.2], All Noetherian modules, semisimple modules, free modules over commutative principal ideal domains, and the Prüfer p-group $\mathbb{Z}_{p^\infty}$ (for any prime p) satisfy the epi-ascending chain condition (epi-acc) on submodules. Below, we list several important properties and results referenced in the preceding paper.*

Proposition 3.76. *[14, Proposition 2.5], Let $M_1, M_2, \ldots, M_n$ be R-modules with epi-acc on submodules, and let*

$$M = \bigoplus_{i=1}^{n} M_i.$$

If M is duo, then it satisfies epi-acc on submodules.

Proposition 3.77. *[14, Proposition 2.7], If S is a simple R-module and M is an R-module that satisfies the epi-acc on submodules, then the direct sum*

$$A = S \oplus M$$

also satisfies the epi-acc on submodules.

Corollary 3.78. *If S is a simple R-module and M is an R-module that satisfies the epi-acc on essential submodules, then the direct sum*

$$A = S \oplus M$$

also satisfies the epi-acc on essential submodules.

Proof. The proof is essentially identical to that of [14, Proposition 2.7]. $\square$

Theorem 3.79. *[14, Theorem 3.2], Let M be a nonzero R-module such that $M^{(N)}$ satisfies epi-acc on submodules. Then:*

  1. Every nonzero submodule of M has a maximal submodule.

2. *If M is finitely generated or Artinian, then M is Noetherian.*

3. *If M is injective, then every projective submodule of M is injective.*

Corollary 3.80. *Let M be a nonzero R-module for which $M^{(N)}$ satisfies the epi-acc on essential submodules. Then the following hold:*

1. *Every nonzero essential submodule of M possesses a maximal submodule.*

2. *If M is finitely generated or essential Artinian, then M is essential Noetherian.*

3. *If M is injective, then every projective submodule of M is also injective.*

Proof. The proof is essentially identical to that of [14, Theorem 3.2].   □

Proposition 3.81. *[14, Proposition 3.4], Let R be a ring. Then:*

1. *If $R^{(N)}$ satisfies the epi-acc on submodules as a right R-module, then R is right Noetherian.*

2. *If $E(R_R)^{(N)}$ satisfies the epi-acc on submodules, then R is a quasi-Frobenius ring.*

Proposition 3.82. *[14, Proposition 3.10], Let R be a left hereditary ring and M a nonzero R-module. If $E(M)^{(N)}$ satisfies the epi-acc on submodules, then M is both semisimple and injective.*

Proposition 3.83. *[14, Proposition 5.5], Let D be a principal left ideal domain. Then we have:*

1. *For every $n \in \mathbb{N}$, the module $D(n)$ satisfies the epi-dcc on submodules.*

2. *For every $n \in \mathbb{N}$, the matrix ring $\mathrm{Mat}_n(D)$ satisfies the left epi-dcc.*

Consequently, any finite direct product of full matrix rings over principal left ideal domains also satisfies the left epi-dcc.

Recall from [14], A ring R is said to be *semiperfect* if $R/J(R)$ is semisimple Artinian and every idempotent in $R/J(R)$ can be lifted to R. An idempotent $e \in R$ is called a *local idempotent* if eRe is a local ring, in which case $_R eR$ is a local module and $J(R)e$ is its unique maximal submodule. A subset A of R is called *left T-nilpotent* if, for every sequence $\{a_1, a_2, a_3, \dots\} \subseteq A$, there exists $n \in \mathbb{N}$ such that

$$a_1 a_2 a_3 \cdots a_{n-1} a_n = 0.$$

For example, $\mathbb{Z}/p^k\mathbb{Z}$ is T-nilpotent as a $\mathbb{Z}/p^k\mathbb{Z}$-module.

A ring R is *left perfect* if $R/J(R)$ is semisimple Artinian and $J(R)$ is left T-nilpotent. Equivalently, R is left perfect if and only if $R/J(R)$ is semisimple Artinian and every nonzero left R-module possesses a maximal submodule.

Proposition 3.84. *[14, Proposition 6.4], Let M be an R-module. If $M^{(\mathbb{N})}$ satisfies the epi-dcc on submodules, then $M^{(\mathbb{N})}$ is* epi-retractable.

Proposition 3.85. *[14, Proposition 6.7], Let R be a left self-injective and left perfect ring that satisfies the left epi-dcc. Then the Jacobson radical $J(R)$ is nilpotent.*

In [25], Jans introduced and investigated co-Noetherian rings, which are rings whose factor modules of co-finitely generated modules remain co-finitely generated, providing characterizations based on injective hulls of simple modules and exploring their fundamental properties and examples. A module M is *co-finitely generated* if for every chain of submodules $\{M_i\}$ such that $\bigcap_i M_i = 0$, there exists an index i such that $M_i = 0$. The ring R is *left co-Noetherian* if every factor module of a co-finitely generated left R-module is itself co-finitely generated. For instance, all finite abelian groups as well as all Artinian modules are co-finitely generated and Z_{p^∞} is a co-Noetherian $\mathbb{Z}$-module. Below, we present several important results and properties that were examined in the preceding paper.

Theorem 3.86. *[25, Theorem 2.1], If R is both left Noetherian and left co-Noetherian, then $J(R)^r = (0)$.*

In [40], the author introduced and studied dual generalizations of the Artinian and Noetherian module conditions, defining Min modules (modules in which every nonzero factor module has a minimal submodule) and Max modules (modules in which every nonzero submodule has a maximal submodule). It explores their structural properties, relationships with Artinian and Noetherian modules, and characterizations via socle and radical behavior, showing that these dual concepts provide a unified framework connecting finitely generated, essential, and torsion-theoretic module properties. Recall from [40], Let M be a Min module. If the socle of M is not essential, then there exists a nonzero submodule A of M such that the direct sum $s(M) + A$ forms an essential submodule of M. The following are some of the key properties and results highlighted in the preceding paper, and is essential for the development of the results.

Proposition 3.87. *[40, Proposition 2.2], For a module M, the following conditions are equivalent:*

1. *The module M is a* Max module*;*

2. *Every nonzero submodule of M has a small radical;*

3. *The upper radical of M is zero, that is, R-rad $M = (0)$;*

4. *There exists a well-ordered decreasing sequence of submodules*

$$M(0) = M \supset M(1) \supset \cdots \supset M(\gamma) = (0),$$

   such that for every ordinal x, the proper inclusion $M(x + 1) \subset M(x)$ implies that $M(x + 1)$ is a maximal submodule of $M(x)$.

In [45], Smith investigates modules that satisfy chain conditions on non-essential submodules, exploring when such modules are uniform, Artinian, or Noetherian. The authors establish characterizations for modules satisfying either the descending chain condition (dcc) or the ascending chain condition (acc) on non-essential submodules, showing their connections to properties like finite Goldie dimension, finitely cogenerated structure, and uniformity. The study further extends these results to rings and commutative modules, offering examples and applications that highlight the distinctions between chain conditions on essential and non-essential submodules. For example, the module $\mathbb{Z}/p^k\mathbb{Z}$ satisfies the ascending and descending chain conditions on non-essential submodules. Recall from [45], a module M is said to have *finite Goldie dimension* if it does not contain a direct sum of infinitely many non-zero submodules. The module M is called *decomposable* if it can be expressed as a direct sum

$$M = M_1 \oplus M_2$$

of two non-zero submodules M_i ($i = 1, 2$). For instance, every finite abelian group has finite Goldie dimension, and, more generally, any module of finite length (i.e., both Artinian and Noetherian) possesses finite Goldie dimension. A submodule K of a module M is termed *(essentially) closed* in M if, for every submodule N of M such that K is essential in N, it follows that $K = N$. The following are some of the key properties and results highlighted in the preceding paper and essential for the development of the results.

Theorem 3.88. *[45, Theorem 1.4], For a module M, the following conditions are equivalent:*

1. *M satisfies the descending chain condition (dcc) on non-essential submodules;*

2. *Every non-essential submodule of M is Artinian;*

3. *Every proper closed submodule of M is Artinian;*

4. *Every decomposable submodule of M is Artinian.*

Corollary 3.89. *[45, Corollary 1.5], Let R be a commutative Noetherian ring. Then an R-module M satisfies the descending chain condition (dcc) on non-essential submodules if and only if M is either uniform or Artinian.*

Proposition 3.90. *[45, Proposition 1.9], Let N be a submodule and K a closed submodule of a module M. If M satisfies the descending (respectively, ascending) chain condition on non-essential submodules, then the same property also holds for the modules N and M/K.*

In [13], Dastanpour et al. introduced and studied modules and rings that satisfy divisibility on chains of submodules or ideals, generalizing the classical ascending and descending chain conditions (acc and dcc). The authors define the notions of ACC_d and DCC_d, where each step in a chain of submodules is related by an endomorphism, and investigate their structural consequences. They show that finitely generated self-projective and self-injective modules with these properties are finite direct sums of uniform modules, and that regular self-injective rings satisfying ACC_d or DCC_d are semisimple Artinian. The paper further explores conditions under which modules or rings with these divisibility properties

satisfy ordinary chain conditions on prime ideals or prime submodules, and concludes that a commutative ring with DCC_d on ideals has Krull dimension at most one, thus connecting divisibility on chains with deep results in module and ring theory. The following are some of the key properties and results highlighted in the preceding paper and essential for the development of the results.

Theorem 3.91. *[13, Theorem 2.1], Let M be an R-module that satisfies the* epi-ascending chain condition (epi-acc) *on submodules. Then every nonzero submodule of M either contains a nonzero Noetherian submodule or possesses a finite uniform dimension. Consequently, every nonzero submodule of M contains at least one uniform submodule.*

Corollary 3.92. *[13, Corollary 2.2], Let M be a nonzero R-module with the* epi-acc *on submodules. Then every nonzero submodule of M contains an essential submodule which is a direct sum of uniform submodules, almost all of which are Noetherian.*

Recall from [13], A nonzero module that is isomorphic to each of its nonzero submodules is called *isosimple.*

Proposition 3.93. *[13, Proposition 2.4], Let M be a nonzero R-module satisfying the* epi-dcc *on submodules. Then every nonzero submodule of M contains an* isosimple *submodule.*

Recall from [13], A ring R is said to satisfy the ACC_d on left ideals if, for every ascending chain

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

of left ideals of R, there exists $x_i \in R$ such that

$$A_i = x_i A_{i+1} \quad \text{for all sufficiently large } i.$$

A ring R is said to satisfy the ACC_d on essential ideals. For example, every PID satisfies ACC_d.

Proposition 3.94. *[13, Proposition 3.3], Let M be an R-module satisfying ACC_d on submodules, and let N be a direct summand of M. Then N also satisfies ACC_d on submodules.*

Theorem 3.95. *[13, Theorem 4.4], Let R be a left self-injective regular ring. If R satisfies the ACC_d on left ideals, then R is semisimple Artinian.*

Recall from [13], a ring R is reversible if $ab = 0$ for $a, b \in R$ implies $ba = 0$. A ring R is called quasi-reversible-over-prime-radical (QRPR for short) if $ab = 0$ for $a, b \in R$ implies $ba \in N(R)$. For example, every commutative ring and every division ring is reversible; furthermore, $\mathbb{Z}/p^k\mathbb{Z}$ is quasi-reversible with respect to its prime radical (p). A ring R is called left morphic if for every $x \in R$, $ann_R(x).r \cong R/Rx$.

Proposition 3.96. *[13, Proposition 5.1], Let R be a ring satisfying ACC_d (respectively, DCC_d) on prime ideals. Then, in each of the following cases, R also satisfies* acc *(respectively,* dcc*) on prime ideals:*

1. *R is a right nonsingular ring with finite left uniform dimension;*

2. *R is quasi-reversible over its prime radical (QRPR);*

3. *R is left morphic.*

In [6], Chaturvedi et al. investigate and characterize a class of modules, called non-small-Noetherian (ns-noetherian) modules, which satisfy the ascending chain condition on non-small submodules, extending earlier studies on modules with finite spanning dimension and exploring their new properties and relationships with Noetherian and hollow modules. The ensuing list outlines several fundamental properties and results drawn from the preceding paper, each of which is crucial to the theoretical framework developed here.

Lemma 3.97. *[6, Lemma 1], Let M be an R-module.*

1. *M is* ns-Noetherian *if and only if every nonempty collection of non-small submodules of M possesses a maximal element.*

2. *M is Noetherian precisely when it is ns-Noetherian and also satisfies the ascending chain condition (acc) on small submodules.*

Recall from [6], A submodule N of a module M is said to be *co-closed* in M (denoted by $N \leq_{cc} M$) if, for every submodule $K \subseteq N$, the condition

$$\frac{N}{K} \leq_s \frac{M}{K}$$

implies that $K = N$. For example, in the module $\mathbb{Z}/p^2\mathbb{Z}$, the submodule $p\mathbb{Z}/p^2\mathbb{Z}$ is co-closed. It is important to note that every nonzero co-closed submodule is necessarily non-small.

Proposition 3.98. *[6, Proposition 3.1], Let N be a submodule of a* ns-Noetherian *module M. Then:*

1. *The quotient module M/N is also ns-Noetherian.*

2. *If N is co-closed in M, then N itself is ns-Noetherian.*

Recall from [6], a submodule K of a module M is called a *supplement* of a submodule N if K is minimal with respect to the property that

$$K + N = M.$$

Moreover, every supplement submodule is necessarily co-closed.

Proposition 3.99. *[6, Proposition 3.2], Let N be a small and Noetherian submodule of a module M. Then M is* ns-Noetherian *if and only if the quotient module M/N is ns-Noetherian.*

Proposition 3.100. *[6, Proposition 3.3], Consider the following assertions for a module M:*

1. *M is* ns-Noetherian.

2. *For every non-small submodule N of M, the quotient module M/N is* Noetherian.

3. *Every* decomposable factor module *of M is* Noetherian.

4. *For every nonzero co-closed submodule C of a module M, the quotient module M/C is* Noetherian.

Recall from [6], an ideal I of a ring R is said to be *irreducible* if there does not exist ideals $I_1 \supsetneq I$ and $I_2 \supsetneq I$ such that

$$I = I_1 \cap I_2.$$

It is called *primary* if, for any $a, b \in R$, the condition $ab \in I$ implies that either $a \in I$ or there exists some $n \in \mathbb{N}$ such that $b^n \in I$.

Furthermore, I is termed *decomposable* if it can be expressed as a finite intersection of primary ideals.

Proposition 3.101. *[6, Proposition 4.1], Let I be a non-small ideal in a commutative* ns-Noetherian *ring R. Then:*

1. *I includes a finite product of prime ideals.*

2. *I can be expressed as the intersection of finitely many irreducible ideals.*

3. *If I is irreducible, then it is also primary.*

4. *Consequently, I is decomposable.*

In [44], Shirali et al. introduced and studied modules and rings that satisfy ascending and descending chain conditions on non-parallel submodules called non-parallel-Noetherian (np-Noetherian) and non-parallel-Artinian (np-Artinian) modules establishing their structural properties, relationships with other chain conditions, and characterizations for the corresponding non-parallel-iso-Noetherian (npi-Noetherian) and non-parallel-iso-Artinian (npi-Artinian) cases. Recall that [44], Two modules N and P are called orthogonal, denoted $N \perp P$, if they do not have any non-zero submodules that are isomorphic. For example, if $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and we take $A = (\mathbb{Z}/p\mathbb{Z}) \oplus 0$ and $B = 0 \oplus (\mathbb{Z}/p\mathbb{Z})$, then $A \cap B = 0$, so A and B are orthogonal submodules of M. Similarly, modules N_1 and N_2 are said to be parallel, written $N_1 \parallel N_2$, if there is no non-zero submodule $V_2 \subseteq N_2$ with $N_1 \perp V_2$, and no non-zero submodule $V_1 \subseteq N_1$ with $N_2 \perp V_1$. For example, let $M = \mathbb{Z} \oplus \mathbb{Z}$. Taking $A = \mathbb{Z} \oplus 0$ and $B = 2\mathbb{Z} \oplus 0$, we see that both are contained in the same direct summand of M; therefore, A and B are parallel submodules. The ensuing list outlines several fundamental properties and results drawn from the preceding paper, each of which is crucial to the theoretical framework developed here.

Theorem 3.102. *[44, Theorem 3.4], Let M be an R-module. The following conditions are equivalent:*

1. *M is a np-Artinian (Noetherian) module;*

2. *Every non-empty collection of non-parallel submodules of M contains a maximal element;*

3. *Every non-empty chain of non-parallel submodules of M possesses a maximal element;*

4. *Every non-parallel submodule of M is Artinian (Noetherian);*

5. *Every proper type submodule of M is Artinian (Noetherian);*

6. *Every orthogonally decomposable submodule of M is Artinian (Noetherian).*

Theorem 3.103. *[44, Theorem 3.5], Every np-Artinian module has a finite type dimension.*

Recall from [44], A module M is said to be finitely embedded (or finitely cogenerated) if its socle is both essential in M and finitely generated.

Theorem 3.104. *[44, Theorem 3.6], A module is finitely embedded if and only if it contains an essential submodule that is Artinian.*

Proposition 3.105. *[44, Proposition 3.16], Let M be a module such that $T(M) \nsubseteq rad(M)$. Then:*

1. *M is np-Noetherian if and only if M is Noetherian;*

2. *M is np-Artinian if and only if M is Artinian.*

In [28], Khalifa studies the structural conditions under which a power series ring of the form $D + XK[[X]]$, where D is an integral domain with quotient field K, is isonoetherian. Building on the work of Facchini and Nazemian, who characterized isonoetherian valuation domains, the author provides necessary and sufficient conditions for $D + XK[[X]]$ to be isonoetherian, showing that this occurs precisely when the ring extension $D \subseteq K$ is u-isonoetherian. The paper further establishes several equivalent formulations involving Noetherian and FC-domain properties, and demonstrates that if D is integrally closed, then $D + XK[[X]]$ is isonoetherian if and only if D is a semi-local principal ideal domain (PID). Through this analysis, the work deepens the understanding of chain conditions up to isomorphism in the context of power series and pullback constructions.

Example 3.106. *Let V be a valuation domain that is Noetherian. Its power series ring $V[[x]]$ is also Noetherian, hence preserves acc on ideals. More generally, if V is isonoetherian, then $V[[x]]$ remains isonoetherian [28].*
In what follows, we summarize several significant properties and results from the above paper that play a central role in the subsequent discussion.

Theorem 3.107. *[28, Theorem 2.1], If the power series ring $A[[X]]$ is isonoetherian, then:*

1. *The sequence $(A_i)_{i \geq 0}$ eventually stabilizes.*

2. *Each A_0-module A_n is isonoetherian for all $n \geq 1$.*

3. *The ring A itself is Noetherian.*

Recall from [28], A ring extension $R \subseteq S$ is called *u-isonoetherian* if, for every ascending chain of R-submodules

$$W_1 \subseteq W_2 \subseteq \cdots,$$

there exists an index n such that for all $i \geq n$, one has

$$W_i = W_n u_i$$

for some invertible element $u_i \in S$.

Proposition 3.108. *[28, Proposition 2.5], If $R \subseteq S$ is a* u-isonoetherian *ring extension and any of the following conditions hold:*

1. *Every invertible element of S lies in R;*

2. *The group of invertible elements of S is finite;*

3. *The ring S is integral over R;*

4. *R is a finite direct product of fields; or*

5. *R is Noetherian and S is not an integral domain;*

then S is a finitely generated R-module.

Recall from [28], Gilmer defined an integral domain D as an *FC-domain* if every chain of distinct overrings of D is finite; equivalently, D is an FC-domain precisely when the set of its overrings satisfies both the ascending and descending chain conditions (acc and dcc).

Theorem 3.109. *[28, Theorem 3.4], Let D be an integral domain with quotient field K and integral closure $\overline{D}$. The following conditions are equivalent:*

1. *The ring $D + XK[[X]]$ is isonoetherian.*

2. *The ring extension $D \subseteq K$ is u-isonoetherian.*

3. *K is strongly affine over D, and D is an almost Dedekind domain.*

4. *D is a Noetherian FC-domain.*

5. *D is a semi-local principal ideal domain (PID) and a finitely generated D-module.*

In [32], Leary investigates the conditions under which modules over Artinian rings are Hopfian or co-Hopfian, providing characterizations for injective modules and establishing equivalences among Hopfian, co-Hopfian, finitely generated, and finite-length properties, particularly in the setting of commutative and principal ideal Artinian rings. An R-module M is said to be finitely co-generated if its injective envelope $E(M)$ can be expressed as a direct sum of the injective envelopes of only finitely many simple R-modules. The following are some of the key properties highlighted in the preceding paper and essential for the development of the results.

Theorem 3.110. *[32, Theorem 3.1] Let E be an injective R-module. Then E is Hopfian if and only if it is co-Hopfian.*

Theorem 3.111. *[32, Theorem 3.5], Let E be an injective R-module. Then the following conditions are equivalent:*

1. *E is Hopfian;*

2. *E is co-Hopfian;*

3. *E is finitely generated;*

4. *E has finite length (i.e., possesses a finite composition series);*

5. *E satisfies the ascending chain condition (is Noetherian);*

6. *E satisfies the descending chain condition (is Artinian);*

7. *E is finitely cogenerated.*

In [10], Contessa introduces and investigates the concept of DICC rings, or rings that satisfy the doubly infinite chain condition (DICC), meaning that no infinite chain of ideals can extend indefinitely in both directions. The study aims to understand the structure and fundamental properties of such rings. Although all Noetherian rings trivially satisfy this condition, the converse is not necessarily true there exist DICC rings that are not Noetherian. For example, let (V, m) be a DVR, and let E denote the injective hull of the residue field $K = V/m$. Then E is naturally a V-module. Consider the ring

$$R = V \oplus E,$$

where addition is defined componentwise and multiplication is given by

$$(u, e)(u', e') = (uu', ue' + u'e) \quad \text{for all } (u, e), (u', e') \in V \oplus E.$$

The nilradical n of R is the ideal $0 \oplus E$, which is not finitely generated. Consequently, R is not Noetherian [10].

Definition 3.112. *[10] A module M is said to satisfy the doubly infinite chain condition if every infinite chain of its submodules $\cdots \subseteq N_{-2} \subseteq N_{-1} \subseteq N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots$ eventually becomes constant either in the increasing direction, the decreasing direction, or in both. A ring which satisfies the DICC condition as a module is called a DICC ring. For example, every finite-dimensional vector space, semisimple, finite-length module satisfies DICC.*

The following are some of the key properties highlighted in the preceding paper and essential for the development of the results.

Lemma 3.113. *[10, Lemma 1], A DICC domain is Noetherian.*

Proposition 3.114. *[10, Proposition 2], Let R be a non-Noetherian DICC ring, and let J denote its nilradical. Then the following statements hold:*

1. *The set of associated prime ideals of J is finite, and each of these primes is maximal.*

2. *The nilradical J is nilpotent; that is, $J^k = 0$ for some positive integer k.*

3. *The ideal J satisfies the descending chain condition (dcc) on ideals.*

Definition 3.115. *[10], A* min/max ideal *of a ring R is a prime ideal that is simultaneously minimal and maximal in* Spec(R).

Theorem 3.116. *[10, Theorem 2], Let S be a non-Noetherian ring with no min/max ideals, and let J denote its nilradical. Then S satisfies the doubly infinite chain condition (DICC) if and only if the following conditions hold:*

1. *The reduced ring S_{red} is Noetherian.*

2. *The nilradical J is nilpotent.*

3. *The ideal J satisfies the descending chain condition (dcc).*

4. *For every element $x \in S \setminus J$, the quotient $J/(Sx \cap J)$ has finite length.*

Theorem 3.117. *[10, Theorem 3], A ring R satisfies the doubly infinite chain condition (DICC) if and only if either R is Noetherian, or R can be expressed as a direct product $R = S \times A$, where S is a ring satisfying the four conditions of Theorem 3.116, and A is an Artinian ring.*

In [12], Contessa extends her earlier work on Doubly Infinite Chain Condition (DICC) rings to develop a unified theory for both rings and modules satisfying this property. A module (and hence a ring) is said to be DICC if any infinite chain of submodules stabilizes either to the right or to the left. Building on her previous results, where a structural characterization of DICC rings was established, Contessa introduces the concept of almost-divisible modules, which serve as a tool for constructing DICC rings and for exploring their connection with Matlis duality. Below, we list several important properties and results referenced in the preceding paper.

Proposition 3.118. *[12, Proposition 1.1], Let R be a non-Noetherian, non-reduced ring whose nilradical J satisfies the descending chain condition (dcc), and whose reduced ring $\bar{R} = R/J$ is Noetherian. Then, for any element $x \in R \setminus J$, the following statements are equivalent:*

1. *The quotient J/xJ has finite length;*

   2. The quotient $J/(Rx \cap J)$ has finite length.

Definition 3.119. *[12], Consider a ring R and an R-module M. We call M almost divisible when the quotient module M/aM possesses finite length for each $a \in R \setminus J$. We denotes $\mathrm{Ass}(M)$ the set of associated prime ideals of a module M.*

Theorem 3.120. *[12, Theorem 1.3], Let R be a reduced Noetherian ring, and let M be an R-module satisfying the descending chain condition (dcc) with infinite length. Then the ring $R \oplus M$ is a DICC ring if and only if M is almost divisible.*

Lemma 3.121. *[12, Lemma 2.1], Let R be a ring and let M be a DICC R-module. Then:*

   1. If $M \neq 0$, then $\mathrm{Ass}(M) \neq \phi$.

   2. The set $\mathrm{Ass}(M)$ is finite.

Theorem 3.122. *[12, Theorem 2.4], Let M be an R-module. Then the following statements are equivalent:*

   (i) M satisfies the DICC condition.

   (ii) There exists a submodule $N \subseteq M$ such that:

      (a) M/N is Noetherian;

      (b) N satisfies the descending chain condition (dcc);

      (c) for all $x \in M \setminus N$, the quotient $N/(Rx \cap N)$ has finite length.

In [11], Contessa investigates R-modules with the descending and ascending chain condition constraints (DICC), exploring their structural properties, classifications, and the relationships between DICC modules and the nature of the underlying ring, particularly focusing on faithful and standardly DICC modules.

Definition 3.123. *[11], A non-reduced ring is said to be weakly DICC if its nilradical is nilpotent and satisfies the descending chain condition (dcc), while its reduced ring is Noetherian. An R-module is called standardly DICC if it is DICC but neither acc nor dcc.*

Below, we list several important lemmas, propositions and theorems referenced in the preceding paper.

Proposition 3.124. *[11, Proposition 1] A dcc R-module M may be regarded as a module over a Noetherian ring.*

Definition 3.125. *A non-reduced ring is termed weakly doubly infinite chain condition when its nilradical is both nilpotent and satisfies the descending chain condition, and its corresponding reduced ring is Noetherian. Similarly, we refer to an R-module as standardly doubly infinite chain condition (SDICC) if it satisfies the doubly infinite chain condition property but fails to satisfy the ascending chain condition and descending chain condition.*

Lemma 3.126. *[11, Lemma 1], Suppose M is an SDICC R-module. For each element $x \in M \setminus N$, the factor module M/Rx is a Noetherian R-module.*

Remark: [11], Let M be a DICC R-module that can be decomposed as the direct sum of two submodules M_1 and M_2. Then, either both M_1 and M_2 are Noetherian, both are Artinian, or one of them possesses finite length.

Theorem 3.127. *[11, Theorem 3], A faithful SDICC R-module with finitely many minimal ideals decomposes as the direct sum of a finite-length module and an SDICC module without minimal ideals.*

Theorem 3.128. *[11, Theorem 2], If M is a faithful SDICC R-module with no min/max ideals, then the following statements hold:*

1. *Min $Supp(M) = \{p\}$;*

2. *$p = $ nilradical of R.*

Theorem 3.129. *[11, Theorem 4], Let M be an arbitrary R-module. Then M is SDICC if and only if there exists an SDICC submodule $M_0 \subseteq M$ such that $M_0 \subseteq N$, M/M_0 is Noetherian, M/N is an essential extension of $M_0/(N \cap M_0)$, and $N/(M_0 \cap N)$ has finite length. Moreover, if M is SDICC, M_0 can be chosen so that*

$$M_0/(N \cap M_0) \cong R/P_0$$

for some finite ideal P_0 of R.

In [26], the authors introduced and studied iso-DICC modules, a generalization of DICC, iso-Noetherian, and iso-Artinian modules, establishing their structural properties, connections with Krull and Goldie dimensions, and extending key results on semiprime and iso-Artinian rings to broader module classes.

Definition 3.130. *[26], An R-module M is called* iso-DICC *if every doubly infinite chain of submodules*

$$\cdots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

stabilizes up to isomorphism; that is, there exists an integer k such that $N_k \cong N_i$ for all $i \geq k$ or $i \leq k$. In particular, a ring R is said to be right iso-DICC *if it is iso-DICC as a right R-module. For example, every cyclic module over a PID, as well as every finite abelian group, is iso-DICC. In what follows, we summarize several significant propositions and theorems from the above paper that play a central role in the subsequent discussion.*

Theorem 3.131. *[26, Theorem 2.4], For an R-module M, the following statements are equivalent:*

1. *M is iso-DICC.*

2. *For every submodule $N \subseteq M$, either N is iso-Artinian or M satisfies iso-acc on submodules containing N.*

3. *Every non-empty set of submodules of M has either an iso-maximal or an iso-minimal element.*

4. *Every non-empty chain of submodules of M has either an iso-maximal or an iso-minimal element.*

Recall from [26], an R-module M is called *essential iso-Noetherian* if every ascending chain of essential submodules

$$M_1 \subseteq M_2 \subseteq \cdots$$

stabilizes up to isomorphism; that is, there exists an integer n such that $M_i \cong M_n$ for all $i \geq n$. For instance, the module $M = \mathbb{Z}_{16}$ provides an example of an essential iso-Noetherian module.

Proposition 3.132. *[26, Proposition 2.6], Let M be a projective iso-simple module and N an iso-DICC module. Then the direct sum $N \oplus M$ is also an iso-DICC module.*

Proposition 3.133. *[26, Proposition 2.10], Consider $R = \prod_{i \in I} R_i$. If R satisfies the right iso-DICC condition, then the index set I must be finite, and every component ring R_i is right iso-DICC.*

Recall from [26], A submodule N of an R-module M is called *fully invariant* if it is preserved under all endomorphisms of M, that is, $f(N) \subseteq N$ for every $f \in \mathrm{End}_R(M)$. The module M is termed a *duo module* if all of its submodules are fully invariant, and a ring R is called a *right duo ring* if the right module R_R satisfies this property. For example, both $\mathbb{Z}/p\mathbb{Z}$ and the Prüfer p-group $\mathbb{Z}_{p^\infty}$ are duo modules over $\mathbb{Z}$.

Proposition 3.134. *[26, Proposition 2.11], Let $M = \bigoplus_{i=1}^n M_i$ be a duo module. Then M is iso-DICC if and only if each M_i is iso-DICC.*

Theorem 3.135. *[26, Theorem 2.14], Suppose M is a finitely generated quasi-projective module with endomorphism ring $S = \mathrm{End}_R(M)$. If M satisfies the iso-DICC condition, then S is a right iso-DICC ring.*

Proposition 3.136. *[26, Proposition 3.3], If I is an annihilator ideal of a semiprime right iso-DICC ring R, then one of the following holds: R satisfies the descending chain condition on annihilator ideals contained in I, or the factor ring R/I satisfies the ascending chain condition on annihilator ideals.*

In [22], Gera investigates the structural properties of modules that satisfy ascending or descending chain conditions on M-cyclic submodules and provides characterizations of such modules in relation to Noetherian and Artinian conditions. A submodule N of M is called M-cyclic if it is isomorphic to M/L for some submodule $L \subseteq M$. It is clear that every M-cyclic submodule is indeed a submodule of M; however, the converse need not hold. For example, $\mathbb{Z}$ is a submodule of $\mathbb{Q}$, but $\mathbb{Z}$ is not a $\mathbb{Q}$-cyclic submodule of $\mathbb{Q}$.

An R-module M is said to be M-Noetherian (respectively, M-Artinian) if it satisfies the ascending chain condition (respectively, descending chain condition) on M-cyclic submodules. That is, for every descending chain of M-cyclic submodules

$$f_1(M) \supseteq f_2(M) \supseteq f_3(M) \supseteq \cdots,$$

there exists an integer n such that

$$f_n(M) = f_{n+1}(M) = f_{n+2}(M) = \cdots.$$

A module M is called iso M-Noetherian (respectively, iso M-Artinian) if every ascending (respectively, descending) chain of M-cyclic submodules

$$f_1(M) \subseteq f_2(M) \subseteq \cdots \quad (\text{or}) \quad f_1(M) \supseteq f_2(M) \supseteq \cdots$$

stabilizes up to isomorphism; that is,

$$f_i(M) \cong f_{i+1}(M) \quad \text{for all sufficiently large } i.$$

It is called mono M-Noetherian (respectively, mono M-Artinian) if in such a chain each

$$f_{i+1}(M) \hookrightarrow f_i(M) \quad (\text{or}) \quad f_i(M) \hookrightarrow f_{i+1}(M))$$

for all sufficiently large i, where $f_i \in \text{End}(M)$. An R-module M is said to be essential M-Noetherian (respectively, essential M-Artinian) if it satisfies the ascending chain condition (respectively, descending chain condition) on essential M-cyclic submodules. That is, for every descending chain of essential M-cyclic submodules

$$f_1(M) \supseteq_e f_2(M) \supseteq_e f_3(M) \supseteq_e \cdots,$$

there exists an integer n such that

$$f_n(M) = f_{n+1}(M) = f_{n+2}(M) = \cdots.$$

For example, simple modules, semisimple modules, modules of finite length, finite-dimensional vector spaces, and finite abelian groups (viewed as $\mathbb{Z}$-modules) are all M-Noetherian (respectively M-Artinian). In particular, for any positive integer n, the module $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is an M-Noetherian (and M-Artinian) $\mathbb{Z}$-module. The following are some of the key properties highlighted in the preceding paper and essential for the development of the results.

Proposition 3.137. *[22, Proposition 1], If M is an iso-Noetherian (respectively, iso-Artinian) module, then every module generated by M is also iso-M-Noetherian (respectively, iso-M-Artinian).*

Proposition 3.138. *[22, Proposition 2], If M is an iso-M-Noetherian (respectively, iso-M-Artinian) module, then every submodule of M also possesses the iso-M-Noetherian (respectively, iso-M-Artinian) property.*

Corollary 3.139. *If M is an essential iso-M-Noetherian (respectively, essential iso-M-Artinian) module, then every essential submodule of M also possesses the essential iso-M-Noetherian (respectively, essential iso-M-Artinian) property.*

Proof. The proof is essentially identical to that of [22, Proposition 2]. □

Proposition 3.140. *[22, Proposition 4], A module M is iso-M-Noetherian (respectively, iso-M-Artinian) if and only if there exists a submodule $K \subseteq M$ such that both K and the quotient module M/K are iso-M-Noetherian (respectively, iso-M-Artinian).*

Corollary 3.141. *A module M is essential iso-M-Noetherian (respectively, essential iso-M-Artinian) if and only if there exists an essential submodule $K \subseteq M$ such that both K and the quotient module M/K are essential iso-M-Noetherian (respectively, essential iso-M-Artinian).*

Proof. The proof is essentially identical to that of [22, Proposition 4].   □

Recall from [22], A ring R is said to be *iso-R-Noetherian* (respectively, *iso-R-Artinian*) if the right R-module R_R is *iso-R-Noetherian* (respectively, *iso-R-Artinian*).

Theorem 3.142. *[22, Theorem 1], If R is an iso-R-Noetherian (respectively, iso-R-Artinian) ring, then the polynomial ring $R[x]$ is also iso-R-Noetherian (respectively, iso-R-Artinian).*

Proposition 3.143. *[22, Proposition 6], If M is an M-Noetherian (respectively, M-Artinian) module of dimension n, then the intersection*

$$\ker f_n \cap f_n(M)$$

is zero.

Corollary 3.144. *If M is an essential M-Noetherian (respectively, essential M-Artinian) module of dimension n, then the intersection*

$$\ker f_n \cap f_n(M)$$

is zero.

Proof. The proof is essentially identical to that of [22, Proposition 6].   □

3.1   Open Problems and Future Directions

Despite significant progress, many open questions remain:

- How do relative injectivity and torsion theories interact with essential chain conditions?

- What roles do Hopfian and co-Hopfian properties play in non-finitely generated modules under chain conditions?

- Can the arithmetic descriptions of projective modules be extended to more general rings?

- How can these ideas be applied in noncommutative algebraic geometry and in homological algebra?

Acknowledgment

References

[1] K. F. U. Ahmed, L. P. Thao and N. V. Sanh, *On semiprime modules with chain conditions*, East–West J. Math. **15** (2013), no. 2, 135–151.

[2] T. Albu and S. T. Rizvi, *Chain conditions on quotient finite dimensional modules*, Comm. Algebra **29** (2001), no. 5, 1909–1928.

[3] T. Albu and M. L. Teply, *The double infinite chain condition and generalized deviations of posets and modules*, Contemp. Math., Vol. 259, Amer. Math. Soc., Providence, RI, 2000, pp. 13–44.

[4] E. P. Armendariz, *Rings with DCC on essential left ideals*, Comm. Algebra **8** (1980), no. 3, 299–308.

[5] P. Aydogdu, A. C. Ozcan and P. F. Smith, *Chain conditions on non-summands*, J. Algebra Appl. **10** (2011), no. 3, 475–489.

[6] A. K. Chaturvedi and N. Kumar, *On modules with chain condition on non-small submodules*, Int. Electron. J. Algebra **33** (2023), 109–124.

[7] A. K. Chaturvedi and S. Prakash, *Some variants of ascending and descending chain conditions*, Comm. Algebra **49** (2021), no. 10, 4324–4333.

[8] H. Chakraborty, P. Nath and R. K. Singh, *Essential Artinian modules and rings*, in *Advances in Pure and Applied Algebra*, De Gruyter, Berlin, 2023.

[9] H. Chakraborty, M. K. Patel and R. K. Singh, *Some aspects on essential Artinian modules and rings*, Asian-Eur. J. Math. **18** (2025).

[10] M. Contessa, *On DICC rings*, J. Algebra **105** (1987), 429–436.

[11] M. Contessa, *On modules with DICC*, J. Algebra **107** (1987), 75–81.

[12] M. Contessa, *On rings and modules with DICC*, J. Algebra **101** (1986), 489–496.

[13] R. Dastanpour and A. Ghorbani, *Divisibility on chains of submodules*, Comm. Algebra **46** (2018), no. 1, 1–15.

[14] R. Dastanpour and A. Ghorbani, *Modules with epimorphisms on chains of submodules*, J. Algebra Appl. **16** (2017), no. 6, 1750101.

[15] L. Diracca and A. Facchini, *Descending chains of modules and Jordan–Hölder theorem*, Semigroup Forum **68** (2004), no. 3, 373–399.

[16] N. V. Dung, D. V. Huynh and R. Wisbauer, *Quasi-injective modules with ACC or DCC on essential submodules*, Comm. Algebra **17** (1989), 252–255.

[17] A. Facchini and Z. Nazemian, *Modules with chain conditions up to isomorphism*, J. Algebra **453** (2016), 578–601.

[18] A. Facchini and Z. Nazemian, *On isonoetherian and isoartinian modules*, in *Model Theory of Modules, Algebras and Categories*, Contemp. Math., Vol. 730, Amer. Math. Soc., Providence, RI, 2019.

[19] C. Faith, *Algebra: Rings, Modules and Categories*, Springer-Verlag, New York–Berlin, 1973.

[20] C. Faith, *Orders in simple Artinian rings*, Trans. Amer. Math. Soc. **114** (1965), 61–64.

[21] C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J. **27** (1966), 179–191.

[22] T. Gera, M. K. Patel and S. C. Gannamaneni, *Chain conditions on M-cyclic submodules*, in *Advances in Pure and Applied Algebra*, De Gruyter, Berlin, 2023.

[23] D. Herbera and P. Příhoda, *Big projective modules over noetherian semilocal rings*, arXiv:0903.2965, 2009.

[24] S. K. Jain, S. R. L. Permouth and S. T. Rizvi, *Continuous rings with ACC on essentials are Artinian*, Proc. Amer. Math. Soc. **108** (1990), no. 3, 583–586.

[25] J. P. Jans, *On co-Noetherian rings*, J. London Math. Soc. (2) **1** (1969), 588–590.

[26] S. M. Javdannezhad, M. Maschizadeh and N. Shirali, *On iso-DICC modules*, Comm. Algebra **53** (2025), no. 1, 52–62.

[27] O. A. S. Karamzadeh and M. Motamedi, *On α-DICC modules*, Comm. Algebra **22** (1994), 1933–1944.

[28] M. Khalifa, *Isonoetherian power series rings*, Comm. Algebra **46** (2018), no. 6, 2451–2458.

[29] F. Kourki and R. Tribak, *On seminoetherian rings and modules*, Comm. Algebra **50** (2022), no. 12, 5200–5216.

[30] T. Y. Lam, *A First Course in Noncommutative Rings*, 2nd ed., Graduate Texts in Mathematics, Vol. 131, Springer-Verlag, New York, 2001.

[31] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, Vol. 189, Springer-Verlag, New York, 1999.

[32] F. C. Leary, *Hopfian and co-Hopfian modules over Artinian rings*, arXiv:2112.01596, 2021.

[33] C. P. Lu, *Modules satisfying ACC on a certain type of colons*, Pacific J. Math. **131** (1988), no. 2, 303–318.

[34] E. Matlis, *Modules with descending chain condition*, Trans. Amer. Math. Soc. **97** (1960), 495–508.

[35] C. Musili, *Introduction to Rings and Modules*, Narosa Publishing House, New Delhi, 1992.

[36] P. Nath, H. Chakraborty and G. Bardhan, *e-Noetherian modules and rings*, in *Advances in Pure and Applied Algebra*, De Gruyter, Berlin, 2023.

[37] B. L. Osofsky, *Chain conditions on essential submodules*, Proc. Amer. Math. Soc. **114** (1992), no. 1, 11–19.

[38] S. S. Page and M. F. Yousif, *Relative injectivity and chain conditions*, Comm. Algebra **17** (1989), no. 4, 899–924.

[39] B. M. Pandeya, A. K. Chaturvedi and A. J. Gupta, *Applications of epi-retractable modules*, Bull. Iranian Math. Soc. **38** (2012), no. 2, 469–477.

[40] R. C. Shock, *Dual generalizations of the Artinian and Noetherian conditions*, Pacific J. Math. **54** (1974), no. 2, 227–235.

[41] R. K. Singh, L. Das, H. Chakraborty, Meyibenla and S. Das, *DCC on essential M-cyclic submodules and R-cyclic ideals*, Palestine J. Math. **14** (2025), no. 2, 1280–1287.

[42] R. K. Singh, M. K. Patel and H. Chakraborty, *A note on essential Noetherian modules and rings*, Proc. Jangjeon Math. Soc. **28** (2025), no. 4, 733–741.

[43] R. K. Singh, M. K. Patel and R. Gumde, *On essential M-Artinian modules*, in *Advances in Mathematical and Computational Sciences*, De Gruyter, Berlin, 2024.

[44] N. Shirali, M. Maschizadeh and S. M. Javdannezhad, *Chain conditions on non-parallel submodules*, Int. Electron. J. Algebra **37** (2025), 249–272.

[45] P. F. Smith and M. R. Vedadi, *Modules with chain conditions on non-essential submodules*, Comm. Algebra **32** (2004), no. 5, 1881–1894.

[46] A. K. Tiwary and B. M. Pandeya, *Modules whose nonzero endomorphisms are monomorphisms,* Algebra and its Applications **91** (1984), 199–203.

[47] R. B. Warfield, Jr. and K. R. Goodearl, *An Introduction to Noncommutative Noetherian Rings*, Cambridge Univ. Press, Cambridge, 2004.

[48] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, New York, 1988.

[49] N. A. Zeyada and M. S. Makki, *Some aspects in Noetherian modules and rings*, AIMS Math. **7** (2022), no. 9, 17019–17025.