Galois J. Algebra **2** (2026), no. 1, pp. 26-49.

Galois Journal of Algebra

https://gja.intrasci.net/

Pseudo-linear maps, an overview

André Leroy*, [1]

ABSTRACT

In this survey, we give a brief account of the history of pseudo-linear maps since their introduction by N. Jacobson in 1937. The (σ, δ)-pseudo-linear transformations are introduced via the modules over a skew polynomial ring. Many classical properties of linear maps are shown to have analogues in the setting of (σ, δ)-pseudo-linear transformations. The relations between these maps and the evaluation of skew polynomials are emphasized. It is shown, in particular, how they are useful when evaluation inside a conjugacy class is considered. The evaluation of Ore polynomials with more than one variable is shown to be related to a sequence of pseudo-linear maps. We also briefly present a recent application that allows to easily answer an open question. Many examples are presented all along the text.

ARTICLE HISTORY

Received: 2 December 2025
Revised: 27 January 2026
Accepted: 29 January 2026
Published: 31 March 2026
Communicated by
Najib Mahdou
https://doi.org/10.65908/gja.2026.8125.1012

KEYWORDS

Pseudo linear maps;
skew polynomial rings;
Evaluations;
differential equations.

MSC

16S36;
12E15;
34M15.

1 Introduction

Linear maps are connected with many branches of pure and applied mathematics. They are important tools in Physics, mechanics,... They are related to matrix theory, classical geometry, algebraic geometry, computer science,... Classically, the study of a linear map T defined on a k vector space V is connected with the polynomial ring $k[x]$. Indeed, T induces a $k[x]$-module structure on V via $x.v = T(v)$ and conversely a $k[x]$-module

*Corresponding author

1. Université d'Artois, Faculté Jean Perrin, Rue Jean Souvraz 62 307 Lens, France.

andre.leroy@univ-artois.fr

structure on V gives rise to a linear map via the action of V. This fruitful relation is reinforced by the use of matrices representing T. Many notions appear like eigenvalues, characteristic polynomials, companion matrices, invariant subspaces, decompositions of V,...

While working with modules equipped with various other structures, we can consider linear maps that respect these additional structures. In this survey, we will look at a ring extension sometimes called Ore extensions over a base ring A and denoted $A[t;\sigma,\delta]$. Let us first say a few words about these rings.

Prompted in particular by various considerations related essentially to the algebraic point of view of differential equations, Øystein Ore introduced in [31] the so-called skew polynomial rings. They encompass both actions of automorphisms and of derivations. Ore considered these polynomials with coefficients in a division ring K and denoted them as $K[t;\sigma,\delta]$ (see below for a complete definition and some examples). The noncommutative aspects of this construction were quickly used to construct examples and counter-examples in ring theory. More recently, the factorization properties of skew polynomial rings have been used in coding theory (cf. [2],[3],[4]) . Later many papers appeared considering both the ring theoretical structure and the more arithmetical properties of these polynomials. The interested reader can consult books such as Jacobson [12],[14], PM Cohn [6] or T.Y. Lam [17], J. Dauns [8]. The base ring was no longer assumed to be a division ring, more variables were considered. Specific ideals, modules were introduced and used to obtain results on the skew polynomial rings. Writing the names of Authors who have been using these polynomials is similar to writing a who's who chapter in noncommutative ring theory.

The topic of this survey is the Pseudo-linear maps. These objects were introduced by N. Jacobson in [11] and are the analogues of linear maps but in the noncommutative setting of Ore extensions. In other words just as linear maps are attached to modules over polynomial rings, the Pseudo-linear transformations are attached to modules over skew polynomial rings. The most famous pseudo-linear maps are perhaps the semi-linear ones (the derivation is the zero map). Indeed, in projective geometry, if the base ring is a field (hence commutative) the collineations correspond to semi-linear maps of the vector space. The semi-linear and pseudo-linear transformations were studied, in particular, by N. Jacobson in [13] [14], by PM Cohn [6], who used these to construct embedded division algebras $K \subseteq L$ such that the right dimension of L as K vector space is finite, but the left dimension of this extension is infinite [7].

Another place where pseudo-linear maps are useful is when considering evaluation of polynomials. In full generality, if we consider evaluations of a polynomial when the base ring A is not assumed to be commutative, the variable t shouldn't be assumed to commute with the constant. So that even with only one variable, say t, we should regard monomials of the form $at, tb, atbt^2c, \ldots$, with $a, b, c \in A$. These have been considered (cf. for examples PM Cohn [7], Fuchs [9]) but we will essentially consider skew polynomial rings as introduced formally by Øystein Ore. In a noncommutative setting evaluation of a polynomial is a delicate matter. Many classical properties have to be reconsidered and amended. Obviously, we have to consider right and left evaluations even if the variable commutes with the scalar. On the other hand, while using variables that do not commute with the scalars, the evaluation should take into account the commutation rules. While studying for instance algebraic theory of differential equations, we need to introduce other concepts to handle the derivations. In this context, the indeterminate should act as a given derivation. Similarly, if we consider an endomorphism of some free module, in this case the action of the variable should reflect the endomorphism. Pushed by these kind of motivation Øysteyen Ore introduced skew polynomials over division rings in 1933 (cf. [31]). The skew polynomial rings denoted by $R = K[t; \sigma, \delta]$ takes into account K is a division ring, σ an automorphism and δ a σ-derivation. They are the right objects that allow to see an automorphism or a derivation as the action of a ring and gives rise to R-modules. Since these Ore extensions are left and right Euclidean (remember σ was supposed to be an automorphism and K a division ring) factorizations are "available" (in fact, there is a "unique factorization" which happens more generally in the frame of 2-firs) and this connects with invariant submodules and decompositions of the ambient free module. In this more rigid setting the roots and their connections with factorizations have been studied in particular by Lam-Leroy (cf. [22], [23],[24],...), we will recall some of their features in this survey.

The modules over skew polynomial rings are given by pseudo-linear maps just as linear maps are attached to modules over usual polynomial rings. Pseudo-linear maps were introduced by Jacobson in 1937 [11]. A particular case of these maps, called semi-linear maps naturally appear in projective geometry. This is natural since they preserve collineation; more explanations will be given after the precise definition of PLT. The relation between PLTs and evaluation of Ore polynomial is very strong. Classically the use of these maps gives rise to a product formula even when the base ring is not a division ring. An attempt to construct some noncommutative algebraic geometry was developed in [30]. This paper develops a systematic and unified framework for studying linear equations over a division ring equipped with a pseudo-linear transformation. The aim is to construct a workable analogue of classical affine algebraic geometry-Zariski topology, morphisms, dimension theory, and Nullstellensatz-within a noncommutative environment where usual commutative tools fail. The theory simultaneously encompasses difference algebra (via ring morphisms) and differential algebra (via derivations), by encoding both in the notion of a pseudo-linear operator.

Recently, the PLT appeared also in connection with multivariable extensions. In this setting they are also connected with evaluations and hence connected with algebraic geometry in a noncommutative frame. Some noncommutative arithmetic could also be considered as connected with PLTs via the evaluation.

Let us sum up what was just said: The study of PLTs is motivated, amongst others, by the following: they appear naturally in projective geometry, they describe the modules over Ore extensions, they are essential tools for the evaluation (and hence to zeroes) of skew polynomials with coefficients in a ring.

Let us now briefly describe the content of this survey.

We will first consider the case of vector spaces over division rings. Many of the classical concepts have analogues in this context. In particular, we will see eigenvalues, roots of polynomials, companion matrices, Wedderburn matrices, Algebraic PLTs,...

Skew polynomials with coefficients in a general ring and the pseudo-linear transformations will be considered and the fruitful connection is described and used to get important facts relating roots of a polynomial in conjugacy classes.

Two kinds of skew polynomial rings with more than one variables are studied. The attached twisted linear maps are used to connect their kernels and the evaluations of the polynomials.

We show the strength of the PLT approach by solving a question raised by Werner [36] in a far more general context than the original question.

We end this survey with some questions and remarks paving the way for future work.

2 One variable

a) **The classical case.**

Let k be a field and V a k-vector space. There is a one-to-one correspondence between k-linear maps $T : V \to V$ and $k[x]$ module structure on V. This correspondence gives insight into the subject. Many concepts show up in this setting like characteristic polynomial, eigenvectors, diagonalisation, decomposition in invariant subspaces... Analogues of these exist in the setting of linear maps over free modules.

A semi-linear map between vector spaces over a field K is additive and "twists" scalar multiplication by a field homomorphism (or automorphism) τ :

$$T(v + w) = T(v) + T(w), \quad T(\lambda v) = \tau(\lambda)T(v)$$

Given a vector space V, the set of all invertible semi-linear transformations $V \to V$ (over all field automorphisms) is the group $\Gamma\mathrm{L}(V)$.

Given a vector space V over $K, \Gamma\ \mathrm{L}(V)$ decomposes as the semidirect product

$$\Gamma\mathrm{L}(V) = \mathrm{GL}(V) \rtimes \mathrm{Aut}(K),$$

The projective geometry of a vector space V, denoted $\mathrm{PG}(V)$, is the lattice of all subspaces of V. Although the typical semi-linear map is not a linear map, it does follow that every semi-linear map $f : V \to W$ induces an order-preserving map $f : \mathrm{PG}(V) \to \mathrm{PG}(W)$. That is, every semi-linear map induces a projectivity. The converse of this observation (except for the projective line) is the fundamental theorem of projective geometry. Thus semi-linear maps are useful because they define the automorphism group of the projective geometry of a vector space.

b) **Skew polynomial rings.**

Let A be a ring, σ an endomorphism of A, and δ a σ-derivation of A (i.e. δ is additive and satisfies $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$)

The skew polynomial ring (a.k.a. Ore extension) and denoted by $R = A[t; \sigma, \delta]$ is a ring whose elements are of the form $\sum_{i=0}^{n} a_i t^i$ with $a_i \in A$. The addition is defined as for usual polynomials and multiplication is defined by distributivity and the rule:

$$ta = \sigma(a)t + \delta(a).$$

Let us mention a few classical examples.

Example 2.1. A will be a ring, σ is an endomorphism of A, and δ a σ-derivation of A.

1. If $A = \mathbb{C}$, σ is the complex conjugation $\delta = 0$ we obtain $R = \mathbb{C}[t; \sigma]$. note that $t^2 + 1$ is a central polynomial and the quotient ring $R/(t^2 + 1)R \cong \mathbb{H}$.
2. If k is a field $k[x]$ is the classical polynomial ring, $\sigma = Id.$ and $\delta = \frac{d}{dx}$ is the usual derivation on $k[x]$, we obtain the Weyl algebra $A_1 = k[x][t; Id., \frac{d}{dx}]$. We have $tx = xt + 1$. If $char(k) = 0$, this ring is simple and if $char(k) = p > 0$ the center of A_1 is $k[x^p, t^p]$.
3. Let $\mathbb{F}_q$ be a finite ring where $q = p^n$ with $n \in \mathbb{N}$ and p a prime integer. If θ is the Frobenius map on $\mathbb{F}_q$ we consider $\mathbb{F}_q[t; \theta]$. This kind of ring is used frequently in coding theory, for instance.
4. The analogue of inner derivation is the inner (σ)-derivation induced by an element $a \in A$. This map is denoted by $\delta_{a,\sigma}$ and defined by $\delta_{a,\sigma}(x) = ax - \sigma(x)a$, for $x \in A$. Notice that such a derivation can "be erased" in the sense that $A[t; \sigma, \delta_{a,\sigma}] = A[t - a; \sigma]$, as one checks easily. Let us also mention that in the case there exists an element c in the center of A which is such that $\sigma(c) - c$ is invertible, then the derivation δ is always inner induced by $(c - \sigma(c))^{-1}\delta(c)$.

One now introduce the evaluation of a skew polynomial.

Let A be a ring, $\sigma \in End(A)$, and δ a σ-derivation of A. For $g(t) \in R = A[t; \sigma, \delta]$ and $a \in A$, we define $g(a)$, the evaluation of g at a, to be the only $b \in A$, such that there exists $q(t) \in R$ with

$$g(t) = q(t)(t - a) + b. \qquad (1)$$

Example 2.2 ([20]). 1. If $g(t) = t$ then $g(b) = b$.

2. If $g(t) = t^2$, then $g(b) = \sigma(b)b + \delta(b)$.
3. If $g(t) = t^3$, implies that $g(b) = \sigma^2(b)\sigma(b)b + (\sigma\delta + \delta\sigma)(b)b + \sigma^2(b)\delta(b) + \delta^2(b)$.

Let us briefly give some classical properties of the ring $K[t; \sigma, \delta]$ when the base ring K is a division ring.

Proposition 2.3. *Let K be a division ring and σ, δ be an endomorphism (not necessarily an automorphism) and a σ-derivation of K. We denote $R = K[t; \sigma, \delta]$. Then the followings hold*

1. *The ring R is a left principal domain.*

2. *The ring R is a right Noetherian domain if and only if σ is an automorphism.*

3. *The ring R is not simple if and only if the derivation is quasi-algebraic, which means that there exists an equation $\sum_{i=0}^{n} a_i(\delta)^i - \delta_{a_0,\sigma^n} = 0$.*

Some of these considerations have been studied for the ring $A[t;\sigma,\delta]$ where A is a ring. One of the features of these extensions is to try to use rings of quotients of the base ring when they are available. This is in particular the case of a (semi-) prime ring where the use of the Martindale quotient ring is useful. The following theorem illustrates this remark (cf. [19]).

Proposition 2.4. *Let R be a prime ring and T its symmetric Martindale quotient ring. For any non-zero two-sided ideal I of $R[t;\sigma,\delta]$ there exists a unique monic invariant polynomial $f_I(t) \in T[t;\sigma,\delta]$ having the following properties*

(1) $\deg f_I(t) = \min\{\deg f(t) \mid f(t) \in I\backslash\{0\}\} = n$ *and every polynomial $g(t) \in I$ of degree n can be written in the form $af_I(t)$ for some $a \in R$.*

(2) $I \subset T[t;\sigma,\delta]f_I(t)$.

Using this we obtain a characterization of the simplicity of the polynomial ring when the base ring is a simple ring.

Corollary 2.5. *Suppose that R is a simple ring. The following conditions are equivalent:*

(1) $R[t;\sigma,\delta]$ is a simple ring.

(2) δ is not a quasi-algebraic σ-derivation of R.

Similar results can be found in [32].

c) **Pseudo-linear maps.**

We introduce the main definition of this survey.

Definition 2.6. *Let A be a ring, σ an endomorphism of A, and δ a σ-derivation of A (i.e. δ is additive and satisfies $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$). Let also V stand for a left A-module.*

An additive map $T : V \longrightarrow V$ such that, for $\alpha \in A$ and $v \in V$,

$$T(\alpha v) = \sigma(\alpha)T(v) + \delta(\alpha)v.$$

is called a (σ,δ) pseudo-linear transformation (or a (σ,δ)-PLT, for short).

The pseudo-linear maps were introduced by Jacobson in 1937 [11].

Example 2.7. 1. Any linear map is an $(Id.,0)$ PLT.

2. The map δ considered as an additive map from ${}_AA$ to ${}_AA$ is itself a (σ,δ)-PLT.

3. If σ is an endomorphism of A and δ is a σ-derivation of the ring A and $a \in A$, the map $T_a : A \to A$ defined by

$$T_a(x) = \sigma(x)a + \delta(x)$$

is easily seen to be a (σ, δ)-pseudo-linear map. This example is the most important one since this PLT will enable us to make the connection with the evaluation of skew polynomials.

4. We can generalize the example (3), as follows: Let $V = A^n$ the row left A-module, and $a \in A$ define $T_a(\underline{v}) = \sigma(\underline{v})a + \delta(\underline{v})$.

5. Any σ semi-linear map is of course a $(\sigma, 0)$ PLT.

Proposition 2.8. *Let A be a ring σ a ring endomorphism of A, and δ a σ-derivation of A. For an additive group $(V, +)$ the following conditions are equivalent:*

(i) V is a left $R = A[t; \sigma, \delta]$-module;

(ii) V is a left A-module and there exists a (σ, δ) pseudo-linear map $T : V \longrightarrow V$;

(iii) There exists a ring homomorphism $\Lambda : R \longrightarrow \mathrm{End}(V, +)$.

Proof. (i) $\Rightarrow$ (2) The pseudo-linear map is simply given by the action of t on V.

(ii) $\Rightarrow$ (iii) The map is simply the one sending a polynomial $\sum a_i t^i$ on $\sum a_i T^i$.

(iii) $\Rightarrow$ (i) This is clear. □

We thus have the following immediate Corollary.

Corollary 2.9. *For any $f, g \in R = A[t; \sigma, \delta]$ and any pseudo-linear transformation T we have: $(fg)(T) = f(T)g(T)$.*

These results are the cornerstones that will enable the generalization of the product formula for Ore polynomial rings defined over rings.

Example 2.10. (1) If $\sigma = id.$ and $\delta = 0$, a pseudo-linear map is an endomorphism of left A-modules. If $\delta = 0$, a pseudo-linear map is usually called a (σ) semi-linear transformation.

(2) If T is a PLT defined on ${}_AV$, V becomes an $(R, End_R(V))$-bimodule. It is often interesting to have a concrete description of the ring $End_R(V)$. For instance, let $a \in A$ and consider the left R-module $V = R/R(t-a)$. The action of t on V is given by $t.(b + R(t-a)) = \sigma(b)t + \delta(b) + R(t-a) = \sigma(b)a + \delta(b) + R(t-a)$. This translates into the following action on A: $t.b = \sigma(b)a + \delta(b)$, for $b \in A$. If we identify $R/R(t-a)$ with A itself, this action corresponds to

$$T_a :_A A \to_A A : T_a(x) = \sigma(x)a + \delta(x).$$

In particular, we get back that $\delta = T_0$ is a PLT.

(3) Extending σ and δ to $M_n(A)$. For $n, l \in \mathbb{N}$ we may also extend component-wise σ and δ to the additive group $V := M_{n\times l}(A)$. Let us denote these maps by S and D respectively. Then S is a σ semi-linear map and D is a (σ,δ)-PLT of the left $M_n(A)$-module V. This generalizes the fact, mentioned in example (2) above, that δ itself is a pseudo-linear transformation on A.

(4) Let ${}_AV_B$ be an (A, B)-bimodule and suppose that σ and δ are an endomorphism and a σ-derivation on A, respectively. If S is a σ semi-linear map and T is a (σ,δ) PLT on ${}_AV$, then for any $b \in B$, the map T_b defined by $T_b(v) = S(v)b + T(v)$, for $v \in V$, is a (σ,δ) pseudo-linear map on V.

(5) Using both Examples (3) and (4) above, we obtain a (σ,δ) pseudo-linear transformation on the set of rectangular matrices $V := M_{n\times l}(A)$ (considered as an ($M_n(A), M_l(A)$)-bimodule) by choosing a square matrix $b \in M_l(A)$ and putting $T_b(v) = S(v)b + D(v)$ where S and D are defined component-wise as in Example (3) and $v \in V$.

In the following Remarks, we attract the attention of the readers on some differences that exist between the world of linear maps and the world of PLTs.

Remark 2.11. *(1) Powers of a PLT don't give a PLT. For instance, if T is a (σ,δ) PLT defined on ${}_AV$, we have the following formula: for $a \in A$ and $v \in V$*

$$T^n(av) = \sum_{i=0}^{n} f_i^n(a)T^i(v),$$

where f_i^n is the sum of the words in σ and δ with i letters σ and $n-i$ letters δ.

(2) If V is a free A-module, we can attach a matrix to any PLT. Although this matrix completely determines T a little care is needed. For instance the derivation δ is a PLT on ${}_AA$ and its matrix in the basis $\{1_A\}$ is just 0 although $\delta \neq 0$.

Proposition 2.12. *Consider $R = A[t;\sigma,\delta]$. and let $a \in A$*

1. *If $a \in A$, the (σ,δ) PLT corresponding to the left R module $R/R(t-a)$ is the map T_a defined on ${}_AA$ by $T_a(x) = \sigma(x)a + \delta(x)$. Moreover $End_R(R/R(t-a))$ is isomorphic to the set of $C_a := \{c \in A \mid T_a(c) = ac\}$ and, for any $f(t) \in R$, $f(T_a)$ is a right C_a-linear map.*

2. *If Σ denotes the set of σ-semi-linear maps defined on a left A-module ${}_AV$ and T is a (σ,δ)-pseudo-linear map defined on V, then the set of all (σ,δ) PLT maps defined on V is given by $T + \Sigma$.*

The next examples provide some applications and extensions of the above Proposition 2.10.

Example 2.13. *1. As we mentioned, the derivation δ is a (σ,δ) PLT defined on A. On the other hand, a σ semi-linear map φ from A to A satisfies $\varphi(x) = \sigma(x)\varphi(1)$ and is thus determined by $a := \varphi(1)$. So if we denote φ_a the σ semi-linear map such* *that* $\varphi(1) = a$*, the set* $\{\varphi_a + \delta \mid a \in A\}$ *gives all the* (σ, δ)*-PLT of* ${}_AA$ *i.e. the PLT* ψ_a *on* ${}_AA$ *are maps defined by* $\psi_a(x) = \sigma(x)a + \delta(x)$*. So that* $\psi_a = T_a$*, where* T_a *is the map defined in Proposition 2.12.*

2. *We can generalize the example (1) above. Suppose* v_0 *is an element of the left* A*-module* V *is such that, for any* $a \in A$*,* $av_0 = 0$ *implies* $a = 0$*. Suppose also that the left* A*-module of* V *generated by* A *is direct summand of* V *i.e. there exists a submodule* W *of* V *such that* $Av_0 \oplus W = V$*. Define a* (σ, δ)*-PLT* T_0 *by* $T_0(v_0) = 0$ *and* $T_0|_W = 0$*. Remark that if* $\delta \neq 0$ *then* T_0 *is not the zero map since we have* $T_0(av_0) = \delta(a)v_0$*. Thanks to the above Proposition 2.12 (2), we conclude that all the PLTs are given by* $T_0 + \Sigma$*, where* Σ *is the set of all* σ*-semi-linear maps defined on* V*.*

3. *Let* $p(t) \in A[t; \sigma, \delta]$ *be a monic polynomial of degree* n*. The module* $R/Rp(t)$*, considered as an* A*-module, is free of rank* n*. The* R*-module structure on* A^n *defined by* $R/Rp(t)$ *corresponds to the* (σ, δ)*-PLT denoted* T_{C_p} *induced by the companion matrix* C_p *of the polynomial* $p(t)$*. To be more explicit, for the row vector* $\underline{v} \in A^n$ *we have*

$$T_{C_p}(\underline{v}) = \sigma(\underline{v})C_p + \delta(\underline{v})$$

.

Remark 2.14. *It is a natural question to try to define maps that resemble linear maps between two different left* A*- modules. This is possible with the help of a secondary linear map. Let* $T :_A V \to_A V$ *be a* (σ, δ)*-PLT, defined on* V *and let* $f :_A V \to_A V$ *be any linear map. We compute* $(T \circ f)(\alpha v) = \sigma(\alpha)(T \circ f)(v) + \delta(\alpha)f(v)$*. Inspired by this we arrive at the following definition.*

Definition 2.15. *Let* ${}_AV_1, {}_AV_2$ *be two left* A*-modules and assume that* f *is* A*-linear. A* (σ, δ, f) *PLT between* ${}_AV_1$ *and* ${}_AV_2$ *is an additive map* $T \in Hom(V_1, V_2)$ *is such that* $T(\alpha v_1) = \sigma(\alpha)T(v_1) + \delta(\alpha)f(v_1)$*, for* $\alpha \in A$*.*

Despite the problems that are mentioned in the above Remarks 2.11, there are still many "standard" behaviors that have analogues. Some of these are present in the following results.

Let us first fix some notations: For a free left A-module V with basis $\beta = \{e_1, \ldots, e_n\}$ and $\varphi \in \mathrm{End}(V, +)$ we write $\varphi(e_i) = \sum_j \varphi_{ij} e_j$ and denote by $\varphi_\beta \in M_n(A)$ the matrix defined by $\varphi_\beta = (\varphi_{ij})$. For a (σ, δ)-PLT $T : V \to V$ is given writing T_β for the matrix of T in the basis β and $\underline{v}$ for the coordinates of a vector $v \in V$ in the same basis, we get the following: $\underline{T(v)} = \sigma(\underline{v})T_\beta + \delta(\underline{v})$.

More generally, we have the following:

Lemma 2.16. *Let* T *be a pseudo-linear transformation defined on a free left* A*-module* V *with basis* $\beta = \{e_1, \ldots, e_n\}$ *and* $f(t) \in R = A[t; \sigma, \delta]$*. Considering* $f(t)$ *as an element of* $M_n(A)[t; \sigma, \delta]$*, we have*

$$f(T)_\beta = f(T_\beta),$$

where the evaluation of matrices is the skew one.

The next Proposition gives some details relating maps and their associated matrices.

Proposition 2.17. *For $i = 1, 2$, let T_i be a (σ, δ) - PLT defined on a free A-module V_i with basis β_i and dimension n_i. Suppose $\varphi \in \mathrm{Hom}_A(V_1, V_2)$ is an A-module homomorphism. Let also $B \in M_{n_1 \times n_2}(A), C_1 \in M_{n_1 \times n_1}(A)$ and $C_2 \in M_{n_2 \times n_2}(A)$ denote matrices representing φ, T_1 and T_2 respectively in the appropriate bases β_1 and β_2. Let ${}_RV_1$ and ${}_RV_2$ be the left R-module structures induced by T_1 and T_2, respectively. The following conditions are equivalent:*

(i) $\varphi \in \mathrm{Hom}_R(V_1, V_2)$;

(ii) $\varphi T_1 = T_2 \varphi$;

(iii) $C_1 B = \sigma(B) C_2 + \delta(B)$;

(iv) $B \in \ker(T_{C_2} - L_{C_1})$ *where* T_{C_2} *(resp.* L_{C_1} *) stands for the pseudo-linear transformation (resp. the left multiplication) induced by* C_2 *(resp.* C_1 *) on* $M_{n_1 \times n_2}(A)$ *considered as a left* $M_{n_1}(A)$*-module.*

The following corollary is an easy particular case of the above Proposition 2.17.

Corollary 2.18. *Let V be a free left A-module of finite rank, $T : V \to V$ be a (σ, δ) PLT, and C_1, C_2 be matrices representing T in the different bases. Then there exists an invertible matrix P such that*

$$C_1 = \sigma(P) C_2 P^{-1} + \delta(P) P^{-1}.$$

Let $p(t) = \sum_{i=0}^{n} a_i t^i$ be a monic polynomial of degree n and consider the left $R = A[t; \sigma, \delta]$ module $V := R/Rp$. It is a free left A-module with basis $\beta := \{\overline{1}, \overline{t}, \ldots, \overline{t^{n-1}}\}$, where $\overline{t^i} = t^i + Rp$ for $i = 1, \ldots, n-1$. In the basis β, the matrix corresponding to left multiplication by t is the usual companion matrix of p denoted by $C(p)$ and defined by

$$C(p) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}$$

Corollary 2.19. *Let $p_1, p_2 \in R = A[t; \sigma, \delta]$ be two monic polynomials of degree $n \geq 1$ with companion matrices $C_1, C_2 \in M_n(A)$. $R/Rp_1 \cong R/Rp_2$ if and only if there exists an invertible matrix B such that $C_1 B = \sigma(B) C_2 + \delta(B)$. In other words the (σ, δ)-PLT determined by C_1 and C_2 are similar.*

Corollary 2.20. *Let $p(t) \in R = A[t; \sigma, \delta]$ be a monic polynomial of degree n and denote by $C = C(p)$ its companion matrix. We have:*

(a) The eigenring $\mathrm{End}_R(R/Rp)$ *is isomorphic to*

$$C_p^{\sigma,\delta} := \{B \in M_n(A) \mid CB = \sigma(B)C + \delta(B\})$$

.

(b) A^n *has an (* $R, C_p^{\sigma,\delta}$ *)-module structure.*

(c) For $f(t) \in R, f(T_p)$ *is a right* $C_p^{\sigma,\delta}$*-morphism. In particular,* $\ker f(T_p)$ *is a right* $C_p^{\sigma,\delta}$*-submodule of* A^n*.*

(d) **PLTs, evaluations and roots**

As mentioned several times, the PLT are strongly connected with evaluations of polynomials.

For $a \in A$, the pseudo-linear transformation on A associated to the left $R = A[t;\sigma,\delta]$-module $V = R/R(t-a)$ is T_a. The equality $f(t)\cdot 1_V = f(a) + R(t-a)$ leads to

$$f(T_a)(1) = f(a)$$

The fact that, for $f(t), g(t) \in R$ we have $fg(T_a) = f(T_a)g(T_a)$ gives

$$(fg)(a) = f(T_a)(g(a)).$$

In particular, for $x \in U(A)$, we have

$$(fx)(a) = f(a^x)x.$$

The set $C^{\sigma,\delta}(a) := \{b \in A \mid ab = \sigma(b)a + \delta(b)\}$ is a ring isomorphic to $\mathrm{End}\,_R V$. We also have

$$\mathrm{Ker}\, f(T_a) = \{x \in A \mid (f(t)x)(a) = 0\}.$$

Since our ring is not commutative we cannot use determinant to define characteristic polynomial of a matrix and have nice theorems relating factorization of this polynomial and invariant subspaces. In the case when the base ring A is a finite dimensional central simple the use of reduced norms gives some analogue of the Cayley-Hamilton theorem (cf. [11]). Here we just mention that with some restrictions we do have similar facts.

Proposition 2.21. *Let* $p(t) \in R = A[t;\sigma,\delta] \subset M_n(A)[t;\sigma,\delta]$ *be a monic polynomial of degree* $n > 1$*. Then the following assertions are equivalent:*

(i) $t \in \mathrm{Idl}(Rp)$ *(i.e.* $pt \in Rp$*);*

(ii) for any $f \in R, f \in Rp$ *if and only if* $f(C(p)) = 0$*;*

(iii) $p(C(p)) = 0$.

The following theorem is a strong generalization of a theorem due to Gordon-Motzkin(Cf. [10]).

Theorem 2.22. *Let* $f(t) \in R = K[t;\sigma,\delta]$ *be a polynomial of degree* n*. Then:*

1) $f(t)$ *has roots in at most* $n(\sigma,\delta)$*-conjugacy classes, say*

$$\{\Delta(a_1), \ldots, \Delta(a_r)\}, \quad r \le n;$$

2) $\sum_{i=1}^r \dim_{C(a_i)} \ker(f(T_{a_i})) \le n$*, where* $C(a_i) := C^{\sigma,\delta}(a_i)$ *for* $1 \le i \le r$*.*

(e) **Algebraic PLTs, Eigenvalues and diagonalisation**

Many classical known facts for linear maps have analogues for PLT. We collect some results from [11] and [17].

Definition 2.23. *Let T be (σ, δ) PLT $T :_A V \to_A V$.*

- *We say that T is algebraic if there exist $n \in \mathbb{N}^*$, $a_0, a_1, \dots, a_n \in A, a_n \neq 0$ such that $a_nT^n + \cdots + a_1T + a_0I = 0$. If $a_n = 1$ we say the T is integral. In this survey paper we will concentrate on the case when V is a left K-vector space, where K is a division ring. We denote by f_T the minimal monic polynomial that is annihilating T*

- *An element $a \in A$ is a left eigenvalue of T if there exists a nonzero $v \in V$ such that $T(v) = \alpha v$. Very often we will consider a finite dimensional vector space. The spectrum of T, denoted Γ consists of left and right eigenvalues.*

Example 2.24. *1. Any linear map that is defined on a finite dimensional vector space is of course algebraic. In more general setting this is untrue.*

2. Any σ-derivation that is algebraic is an example of algebraic PLT (on ${}_AV =_A A$). The (quasi) algebraicity of δ is strongly connected with the simplicity of $R = A[t; \sigma, \delta]$ (cf. [27]).

Let us first mention some basic properties of eigenvalues and eigenvectors.

Proposition 2.25. *Let $T : V \to V$ be an (σ, δ) PLT and $\alpha \in A, v \in V$ be an eigenvalue and its corresponding eigenvector : $T(v) = \alpha v$. Then :*

- For any $g(t) \in R = A[t; \sigma, \delta]$ we have $g(T)(v) = g(\alpha)v$.

- For $\beta \in A\backslash\{0\}$ and for any $g(t) \in R = A[t; \sigma, \delta]$, we have $g(T)(\beta v) = g(T_\alpha)(\beta)v$.

- If $\dim_K V = n$ and $\underline{e} = (e_1, \dots, e_n)$ is a basis of V, writing $v = \sum \alpha_i e_i$ and $\underline{v} = (\alpha_1, \dots, \alpha_n)$ we have $\sigma(\underline{v})M_{\underline{e}}(T) + \delta(\underline{v}) = \alpha\underline{v}$.

For the remainder of this section, we will assume, for simplicity, that $A = K$ is a division ring. With the help of Martindale quotient ring, we certainly could develop most of the following in the frame of a prime ring. But to avoid some technicalities and keep the essence of the topic clear, it seems preferable to concentrate on the case of a division ring. We remark that T is algebraic exactly when the ring morphism $R = A[t; \sigma, \delta] \stackrel{\varphi}{\to} End(V, +)$ given by $\varphi(f(t)) = f(T)$ has a nonzero kernel. In particular, f_T is invariant and satisfies $f_TR = Rf_T$.

The following results can be found in[17]. We just mention some of the main results.

Proposition 2.26. *Let K be a division ring and σ, δ be an endomorphism of K and a σ-derivation δ defined on K. Suppose that T is an algebraic (σ, δ)-PLT defined on ${}_KV$ and let $f_T(t) \in R = K[t; \sigma, \delta]$ be its minimal polynomial. The following are equivalent:*

(i) $\alpha \in K$ is an eigenvalue for T.

(ii) $t-\alpha$ *divides on the right the polynomial* $f_T(t)$ *in* $R=K[t;\sigma,\delta]$.

(iii) $t-\alpha$ *divides on the left the polynomial* $f_T(t)$ *in* $R=K[t;\sigma,\delta]$.

This Proposition implies that the set $\Gamma\subset K$ of eigenvalues of T is algebraic when T is algebraic. Moreover, if we denote f_Γ its minimal polynomial, we have that f_Γ divides f_T (on the right and on the left). We decompose the set of eigenvalues of T into conjugacy classes and write $\Gamma=\cup_{i=1}^r\Gamma_i$, where $r\leq degf_T$. The annihilating polynomials of these conjugacy classes give factors of F_T.

Remark 2.27. Clearly, $\delta=T_0$ is a (σ,δ) map such that $M_e(T)=0$, where the base $e=\{1\}$. So that $M_e(T)$ is algebraic doesn't imply that T is algebraic. Nevertheless, we have the following Proposition.

Proposition 2.28. *Let* T *be a* (σ,δ)*-PLT defined on a finite dimensional left* K*-vector space* V *with basis* β. T *is algebraic if and only if the conjugacy class determined by the matrix* T_β *is algebraic.*

It might be worth to point out explicitly the following particular case.

Corollary 2.29. *Let* a *be an element of a division ring* K. *The* (σ,δ)*-PLT* T_a *is algebraic if and only there exists a polynomial* $f(t)\in R=K[t;\sigma,\delta]$ *such that* $f(a^x)=0$*, i.e. the conjugacy class determined by the element* a *is algebraic.*

Let us now mention the following result, connecting algebraicity of a PLT and structural properties of the ring $K[t;\sigma,\delta]$. This can be found in [17].

Proposition 2.30. *Let* K *be a division ring and* σ, δ *be an endomorphism (not necessarily an automorphism) and a* σ*-derivation of* K. *We denote* $R=K[t;\sigma,\delta]$. *Then the followings hold*

1. R *is left primitive,*
2. R *is right primitive,*
3. *there exists* $V\xrightarrow{T}V$ *a* (σ,δ)*-PLT that is not algebraic and* $dim_KV<\infty$.

Definition 2.31. *A PLT* $T:_KV\to_KV$ *is diagonalizable if there exists a basis* B *of* V *such that matrix representing* T *in this basis is diagonal.*

The following Theorem (cf. [17]) shows that many classical Theorems have analogues in our situation.

Theorem 2.32. . *Let* T *be an algebraic* (σ,δ) *PLT on a left* K*-vector space* V *such that* $\dim V=n$. *Let* f_T *be its minimal monic polynomial and* $\Gamma=\cup_{i=1}^r\Delta^{\sigma,\delta}(\gamma_i)$ *the set of eigenvalues of* T. *Then the following assertions are equivalent :*

(i) T *is diagonalizable,*

(ii) there exists $\underline{e}$ *a basis of* V *such that* $M_{\underline{e}}(T)$ *is diagonal (in other words* $\Delta(T)$ *contains a diagonal matrix),*

(iii) there exists $\underline{e}$ a basis of V and $\{\delta_1, \ldots, \delta_n\} \subset K$ such that $T_{M_{\underline{e}}(T)} = T_{\delta_1} \oplus \cdots \oplus T_{\delta_n}$ where $\oplus_{i=1}^n T_{\delta_1} : K^n \longrightarrow K^n : (\alpha_1, \ldots, \alpha_n) \longmapsto (T_{\delta_1}(\alpha_1), \ldots, T_{\delta_n}(\alpha_n))$,

(iv) $V_\Gamma = V$, where V_Γ is the vector space generated by the eigenvectors of T,

(v) $f_\Gamma = f_T$, where f_Γ is the minimal polynomial of Γ,

(vi) $\deg f_\Gamma = \deg f_T$,

(vii) $f_T = \prod_{i=1}^r f_{\Gamma_i}$, where f_{Γ_i} is the minimal polynomial of $\Gamma_i = \Delta^{\sigma,\delta}(\gamma_i)$;

(viii) $\sum_{i=1}^n \left[K : C^{\sigma,\delta}(\gamma_i)\right]_{right} = \deg f_T$ where $C^{\sigma,\delta}(\gamma_i) = \{x \in K^* \mid \gamma_i^x = \gamma_i\} \cup \{0\}$ is a subdivision ring of K.

Theorem 2.33. *Let p be a prime number and $\mathbb{F}_q$ be the finite field with $q = p^n$ elements. Denote by θ the Frobenius automorphism. Then:*

a) There are p distinct θ-conjugacy classes in $\mathbb{F}_q$.

b) $C^\theta(0) = \mathbb{F}_q$ and, for $0 \neq a \in \mathbb{F}_q$, we have $C^\theta(a) = \mathbb{F}_p$.

c) In $\mathbb{F}_q[t;\theta]$, the least left common multiple of all the elements of the form $t - a$ for $a \in \mathbb{F}_q$ is the polynomial $G(t) := t^{(p-1)n+1} - t$. In other words, $G(t) \in \mathbb{F}_q[t;\theta]$ is of minimal degree such that $G(a) = 0$ for all $a \in \mathbb{F}_q$.

d) The polynomial $G(t)$ obtained in c) above is invariant, i.e. $RG(t) = G(t)R$.

The next proposition gives some behavior of a PLT corresponding to some special polynomials. The first kind of polynomials appears naturally when considering ring morphisms between Ore extensions over a common base ring.

Definition 2.34. *Let σ, σ' be two endomorphisms of a ring A and δ, δ' be σ and σ' derivations respectively.*

1. *A polynomial $p(t) \in A[t, \sigma, \delta]$ is an (σ', δ') CV polynomial, if for any $a \in A$ we have $p(t)a = \sigma'(a)p(t) + \delta'(a)$.*
2. *A $(\sigma', 0)$ is called a semi-invariant polynomial.*
3. *If $p(t) \in R$ is such that we $p(t)R \subseteq Rp(t)$, we say that $p(t)$ is right invariant.*

The following Theorem is classical.

Theorem 2.35. *The ring $R = K[t;\sigma,\delta]$, where K is a division ring, is not simple if and only if there exists a semi-invariant polynomial of nonzero degree.*

There are strong connections between the set of CV polynomials, semi-invariant polynomials, and invariant polynomials. We refer the interested reader to the papers [23], [24], and [27].

Proposition 2.36. *Let $T : V \to V$ be a (σ, δ)-PLT and let $\varphi : R' = K[t';\sigma',\delta'] \to R = K[t;\sigma,\delta]$ be the K-ring homomorphism defined by a (σ', δ') c.v. polynomial $p(t) \in R$. The R'-module structure induced on V via φ by T corresponds to the R'-module structure* *given by the (σ', δ') PLT $p(T)$. In particular if $g(t') \in R'$ we have, for $A \in M_n(K)$, that the following (σ', δ')-PLT on K^n coincide:*

$$g(p)(T_A) = g\left(T_{p(A)}\right)$$

and for $A \in M_n(K), P \in GL_n(K)$

$$g(p)\left(A^P\right) = g\left(p(A)^P\right)$$

where $g(p) \in R, T_A$ and $T_{p(A)}$ stand for a (σ, δ) and (σ', δ') PLT respectively, $A^P = \sigma(P)AP^{-1} + \delta(P)P^{-1}$ and $p(A)^P = \sigma'(P)p(A)P^{-1} + \delta'(P)P^{-1}$.

Proof. A detailed proof can be found in Lemma 3.3 of [17]. To give a flavor of the topic, we prove here the first formula. The (σ, δ)-PLT $T : V \to V$gives a left R-module structure on V via $t.v = T(v)$. Since $\varphi : R' \to R$ is a ring homomorphism of rings, V inherits a left R' module structure and we have, for $v \in V$, $t'.v = \varphi(t').v = p(t).v = p(T)(v)$. This shows that the R' module structure of V is given by $p(T)$. If $A \in M_n(K)$ and T_A is a (σ, δ)-PLTdefined on $V = K^n$, we have just seen that $p(T_A)$ is the (σ', δ') associated to the corresponding action of t' on K^n. Using Lemma 2.16, we then have

$$g(p)(T_A) = g(p)(t.) = g(p(t.)) = g(t'.) = g(T_{p(A)}),$$

as desired. □

3 Multivariables extensions

a) **Iterated Ore extensions, evaluation, PMT.**

In what follows, we consider an iterated Ore extension. We start with a ring A, $\sigma_1 \in End(A)$ and δ_1 is a σ_1-derivation on A. We then construct the ring $R_0 = A = S_0$ and $R_1 = S_1 = A\left[t_1; \sigma_1, \delta_1\right]$. We then assume that $\sigma_2 \in End(R_1)$ and δ_2 is a σ_2-derivation on R_1 such that $\sigma_2(A) \subseteq A$ and $\delta_2(A) \subseteq A$ and put $S_2 = A[t_2; \sigma_2, \delta_2] \subseteq R_2 = R_1\left[t_2; \sigma_2, \delta_2\right] = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right]$. We continue constructing an iterated Ore extension (For any $1 \le i \le n$, we assume that $\sigma_i(A) \subseteq A$ and $\delta_i(A_i) \subseteq A$):

$S_3 = A[t_3; \sigma_3, \delta_3] \subseteq R_3 = R_2\left[t_3; \sigma_3, \delta_3\right] = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right]\left[t_3; \sigma_3, \delta_3\right]$, and finally

$S_n = A[t_n; \sigma_n, \delta_n]$ a subset of

$$R = R_n = R_{n-1}\left[t_n; \sigma_n, \delta_n\right] = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right] \cdots \left[t_n; \sigma_n, \delta_n\right].$$

We also define, for $1 \le i < j \le n, p_{i,j} = \sigma_j\left(t_i\right) \in R_{j-1}$ and $q_{i,j} = \delta_j\left(t_i\right) \in R_{j-1}$. For any $1 \le i \le n$, we assume that $\sigma_i(A) \subseteq A$ and $\delta_i(A_i) \subseteq A$.

The following definition formalizes the idea that to evaluate an iterated skew polynomial, you first evaluate the last variables t_n and then continue evaluating the previous ones.

Definition 3.1. *For $f(t_1,\ldots,t_n) \in R = A[t_1;\sigma_1,\delta_1][t_2;\sigma_2,\delta_2]\cdots[t_n;\sigma_n,\delta_n]$, we define the evaluation of $f(t_1,t_2,\ldots,t_n)$ at a point $(a_1,\ldots,a_n) \in A^n$, denoted by $f(a_1,\ldots,a_n)$, as the representative in A of $f(t_1,t_2,\ldots,t_n)$ modulo*

$$I_n(a_1,\ldots,a_n) = R_1(t_1-a_1)+\cdots+R_{n-1}(t_{n-1}-a_{n-1})+R(t_n-a_n),$$

where, for each $1\le i\le n, R_i$ stands for $R_i = A[t_1;\sigma_1,\delta_1]\cdots[t_i;\sigma_i,\delta_i]$.

Remark 3.2. When evaluating a polynomial, we must first write it such that in each of its monomials the variables appear in the precise order $t_1,t_2,\ldots,t_n$ (from left to right). In other words, before evaluating a polynomial, we must write the polynomial as a sum of monomials of the form $t_1^{l_1}t_2^{l_2}\cdots t_n^{l_n}$. So if we evaluate $t_1t_2 \in A[t_1,t_2]$ at $(\ a,b\ )$ one obtains $(t_1t_2)(a,b)=ba$. This might look very strange, but regarding evaluation in terms of "operators" via the right multiplication by b followed by the right multiplication by a this evaluation looks perfectly fine and the apparent awkwardness disappears.

Let us now mention concrete examples.

Example 3.3. (1) Let $A_1(k)=k[X]\left[Y;id.,\frac{d}{dX}\right]$ and $(a,b)\in k^2$. Then - $YX = XY+1 = X(Y-b)+bX+1 = X(Y-b)+b(X-a)+ba+1$, and hence $(YX)(a,b)=ba+1$.

- $YX^2 = X^2Y+2X = X^2(Y-b)+bX^2+2X = X^2(Y-b)+bX(X-a)+baX+2(X-a)+2a$, and hence $\left(YX^2\right)(a,b)=ba^2+2a$.

- $Y^2X = XY^2+2Y = XY(Y-b)+bX(Y-b)+bXb+2(Y-b)+2b$, and therefore $\left(Y^2X\right)(a,b)=b^2a+2b$.

(2) Consider the double Ore extension $R=\mathbb{F}_q[t_1;\theta]\left[t_2;\bar\theta\right]$, where $q=p^n$, $\bar\theta(a)=a^p$, and $\bar\theta(t_1)=t_1$. A polynomial $p(t_1,t_2)\in R$ can be written as $p(t_1,t_2)=\sum_{i=0}^n p_i(t_1)t_2^i = \sum_{i,j}a_{i,j}t_1^jt_2^i$. For every $s\ge 1$, one can check $N_s(a)=\theta^{s-1}(a)\ldots\theta(a)a = a^{\frac{p^s-1}{p-1}}$, this leads to

$$p(t_1,t_2)(a,b)=\sum_{i,j}a_{i,j}\theta^j(N_i(b))N_j(a)=\sum_{i,j}a_{i,j}b^{\frac{\left(p^i-1\right)p^j}{p-1}}a^{\frac{p^i-1}{p-1}}.$$

We now try to define analogues of PLT in the multivariable settings. We certainly would like to connect the pseudo multivariable setting to evaluation of polynomials in $R = R[x;\sigma][t_1,\sigma_1,\delta_1]\ldots[t_n\sigma_n,\delta_n]$ and to left R-modules. This last objective quickly leads us to the following definition.

Definition 3.4. *Let A be a ring, $R=A[t_1;\sigma_1,\delta_1]\cdots[t_n;\sigma_n,\delta_n]$ be an iterated Ore extension, ${}_AV$ a left A-module, and $(T_1,\ldots,T_n)$ be a sequence of maps in* $\mathrm{End}(V,+)$. *such that for each $1\le i\le n, T_i$ is a (σ_i,δ_i)-PLT of ${}_AV$. The sequence $(T_1,\ldots,T_n)$ is called good if $({}_AV,T_1\ )$ gives a ${}_{R_1}V$ structure on V, and T_2 is a $(\ \sigma_2,\delta_2\ )$- PLT on ${}_{R_1}V$ so that $(\ {}_{R_1}V,T_2\ )$ defines an ${}_{R_2}V$ structure on V, and inductively, for any $1\le i<n, T_{i+1}$ is a $(\ \sigma_i,\delta_i\ )$-PLT on ${}_{R_i}V$-structure which leads to an R_{i+1} modules stucture on V.*

We immediately remark that a good sequence $(T_1,\ldots,T_n)$ of PLT defined on a left A-module structure ${}_AV$ gives rise to a left R-module structure on V via $t_i.v=T_i(v)$, for $1\le i\le n$.

Example 3.5. One of the most important examples of sequences of PLT comes from the evaluation maps. Let $a = (a_1, \ldots, a_n) \in A^n$ and consider the following iterated Ore extension

$$R = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right] \cdots \left[t_n; \sigma_n, \delta_n\right].$$

For $1 \leq i \leq n$, we define the map $T_i : R_{i-1} \to R_{i-1}$ by $T_i(x) = \sigma_i(x)a_i + \delta_i(x)$ for all $x \in R_{i-1}$. This sequence of PLTs defined on A corresponds to a left R-module structure on A.

We collect in the next Theorem the results showing that our good sequences are exactly what was required.

Theorem 3.6. *Let A be a ring and $R = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right] \cdots \left[t_n; \sigma_n, \delta_n\right]$ an iterated Ore extension on A. For $a = (a_1, \ldots, a_n) \in A^n$, we let $T_i = T_{a_i}$ be the PLT on A defined in the above example and $f = f\left(t_1, \ldots, t_n\right) \in R$. Then the following statements hold.*

(1) For any $x \in A$, we have $\left(fx\right)\left(a_1, \ldots, a_n\right) = f\left(T_{a_1}, \ldots, T_{a_n}\right)(x)$.

(2) We have $f\left(a_1, \ldots, a_n\right) = f\left(T_{a_1}, \ldots, T_{a_n}\right)(1)$.

(3) For any $x \in U(A)$, we have

$$f\left(T_{a_1}, \ldots, T_{a_n}\right)(x) = \left(fx\right)\left(a_1, \ldots, a_n\right) = f\left(a_1^x, \ldots, a_n^x\right)x,$$

where for each $i \in \{1, \ldots, n\}$, $a_i^x = \sigma_i(x)a_ix^{-1} + \delta_i(x)x^{-1}$.

The above Theorem 3.6 shows the connection between evaluation and pseudo multi-linear maps. It is natural to wonder about the existence of a product formula. This the objective of the next result but to state it we need to introduce another way of defining the evaluation.

We evaluate $f\left(t_1, \ldots, t_n\right) \in R = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right] \cdots \left[t_n; \sigma_n, \delta_n\right]$ at the point $(a_1, a_2, \ldots, a_n) \in A^n$ by considering the element of A representing f in the quotient R/I, where $I = R\left(t - a_1\right) + R\left(t - a_2\right) + \cdots + R\left(t - a_n\right)$. It arrives frequently that $I = R$ and this new evaluation is then not a good one. We say that a point $(a_1, \ldots, a_n) \in A^n$ is good if we have $I_n = \sum_{i=1}^{n} R_i(t_i - a_i) = I$. The next proposition will compare the two evaluations by comparing $I_n = R_1\left(t_1 - a_1\right) + \cdots + R_{n-1}\left(t_{n-1} - a_{n-1}\right) + R\left(t_n - a_n\right)$ and I. It will show that a point $(a_1, \ldots, a_n) \in A^n$ is good if and only if the sequence $(T_{a_1}, \ldots, T_{a_n})$ is a good sequence.

Theorem 3.7. *Let A be a ring and $R = A\left[t_1; \sigma_1, \delta_1\right]\left[t_2; \sigma_2, \delta_2\right] \cdots \left[t_n; \sigma_n, \delta_n\right]$. We consider $(a_1, a_2, \ldots, a_n) \in A^n$ and put*

$$I = R\left(t - a_1\right) + R\left(t - a_2\right) + \cdots + R\left(t - a_n\right),$$

and

$$I_n = R_1\left(t_1 - a_1\right) + \cdots + R_{n-1}\left(t_{n-1} - a_{n-1}\right) + R\left(t_n - a_n\right),$$

where, for each $1 \leq i \leq n, R_i = A\left[t_1; \sigma_1, \delta_1\right] \cdots \left[t_i; \sigma_i, \delta_i\right]$. With these notations, the following statements are equivalent:

(1) $I_n = I$ *(and hence the* 2 *evaluations coincide);*

(2) $R\,(t_i - a_i) \subseteq I_n$*;*

(3) $I \neq R$*;*

(4) For $1 \leq i < j \leq n$*, we have* $t_j\,(t_i - a_i) \in I_n$*;*

(5) For $1 \leq i < j \leq n$*, we have* $\sigma_j\,(t_i - a_i)\,a_j + \delta_j\,(t_i - a_i) \in I_n$*;*

(6) For $1 \leq i < j \leq n$*, we have* $(t_j t_i)\,(a_1, \ldots, a_n) = \sigma_j\,(a_i)\,a_j + \delta_j\,(a_i)$*;*

(7) For all $f, g \in R$*, we have*

$$(fg)\,(a_1, a_2, \ldots, a_n) = (f\,(T_{a_1}, T_{a_2}, \ldots, T_{a_n}) \circ g\,(T_{a_1}, T_{a_2}, \ldots, T_{a_n}))\,(1);$$

(8) The sequence ($T_{a_1}, T_{a_2}, \ldots, T_{a_n}$ *) of PLT on* A *is good;*

(9) The map $\psi : R = A\,[t_1; \sigma_1, \delta_1]\,[t_2; \sigma_2, \delta_2] \cdots [t_n; \sigma_n, \delta_n] \longrightarrow \mathrm{End}(A, +)$ *defined by* $\psi\,(f\,(t_1, \ldots, t_n)) = f\,(T_{a_1}, \ldots, T_{a_n})$ *is a ring homomorphism.*

Notice that the point (7) above shows that we cannot expect the product formula for all points.

Let us now give some examples related to good points.

Example 3.8. 1. In the Weyl algebra $R = A_1(K) = K[t_1][t_2; id, \frac{d}{dt_1}]$ for the point $(0,0)$ since we then have $t_2t_1 - t_1t_2 = 1$ and hence $Rt_1 + Rt_2 = R$. This is not the case with our evaluation since, for instance, $t_2t_1 = t_1t_2 + 1$, so that $t_2t_1 + I_2(0,0) = 1 + I_2(0,0)$ and hence the evaluation of t_2t_1 at $(0,0)$ is just 1.

2) In fact, it is quite often the case that $I = R$, even if we are using Ore polynomials with zero derivations. Consider for instance the Ore extension $R = K[t_1; \sigma_1][t_2; \sigma_2]$ where K is a field and σ_2 is an endomorphism of $K[t_1; \sigma_1]$ such that $\sigma_2(t_1) = t_1$. It is easy to check that for any $(a_1, a_2) \in K^2$ we have $(t_2 - \sigma_1(a_2))(t_1 - a_1) + (-t_1 + \sigma_2(a_1))(t_2 - a_2) = \sigma_1(a_2)a_1 - \sigma_2(a_1)a_2$. So that if $\sigma_1(a_2)a_1 - \sigma_2(a_1)a_2 \neq 0$, then the left ideal $I(a_1, a_2) = R$. This shows that very often the evaluation modulo I turns out to be trivial. Once again, this is not the case with our evaluation, since we have that $t_2(t_1 - a_1)$ is represented by $\sigma_1(a_2)a_1 - \sigma_2(a_1)a_2$ modulo $I_2(a_1, a_2)$.

b) **Another kind of skew multivariables polynomials.**

The following definition was introduced by Matinez-Peñas and Kschischang (cf. [29]). An earlier version of this construction was given by Helena Weixler-Kreindler [36].

Definition 3.9. *Consider a ring* A, n *free variables* $t_1, \ldots, t_n, \underline{\sigma} : A \to M_n(A)$ *a ring homomorphism, and a sequence of* n *additive maps* $\delta_1, \ldots, \delta_n$*. We denote by* M *the free monoid generated by the variables* $\{t_1, \ldots, t_n\}$ *and by* $B = A[\underline{t}; \underline{\sigma}, \underline{\delta}]$ *the set of polynomials of the form* $\sum_{m \in M} \alpha_m m$*, where* $\alpha_m \in A$ *and* $m \in M$*. On this set, we define the natural addition and we introduce a multiplication based on the concatenation in* M *and on the* *following commutation rules:*

$$\forall 1 \leq i \leq n, \forall a \in A, \quad t_i a = \sum_{j=1}^{n} \sigma(a)_{ij} t_j + \delta_i(a).$$

Example 3.10. (1) In the case $n = 1$ the above definition gives back the classical Ore extension.

(2) Let $B = A[\underline{t}; \underline{\sigma}, \underline{\delta}]$ be a Multivariable Ore polynomial ring. Of course, if A is a central simple algebra the classical Skolem-Noether Theorem guarantees that there exists an invertible matrix $U \in M_n(A)$ such that $\sigma(a) = U\iota(a)U^{-1} = U\,(aI_n)\,U^{-1}$ for all $a \in A$, where $\iota : A \to M_n(A)$ is defined by $\iota(a) = aI_n$. In this case we have that $B = A[\underline{t}; \underline{\sigma}, \underline{\delta}] = A[U^{-1}t, \iota, U^{-1}\delta]$.

(3) Let A be a ring, $\alpha, \beta \in \mathrm{End}(A)$, and γ be an (α, β)-derivation (i.e. $\gamma \in \mathrm{End}(A, +)$ and, for any $a, b \in A$ we have $\gamma(ab) = \alpha(a)\gamma(b) + \gamma(a)\beta(b)$). We can check that the map

$$\underline{\sigma} : A \longrightarrow M_2(A) : a \mapsto \begin{pmatrix} \alpha(a) & \gamma(a) \\ 0 & \beta(a) \end{pmatrix}$$

is a homomorphism of rings. If $x \in A$ we can define an (α, β)-derivation γ via $\gamma(a) = x\beta(a) - \alpha(a)x$. Such an (α, β)-derivation is called inner. The map $\underline{\sigma}$ above gives rise to the extension $A\left[(t_1, t_2)^t\,; \sigma\right]$. Let us give an easy example of an α, β-derivation where both α and β are not inner. Consider

Let k be a field and choose $q, r \in k^{\times}$with $q \neq r$ (and hence $q - r \neq 0$). Put $A = k[x]$ and define automorphisms $\alpha, \beta : A \to A$ by

$$\alpha(f)(x) = f(qx), \quad \beta(f)(x) = f(rx).$$

Define $\delta : A \to A$ by the divided-difference operator

$$\delta(f)(x) = \frac{f(qx) - f(rx)}{(q - r)x}.$$

One can check that δ is indeed an (α, β) non inner derivation.

Let us notice that in the case of an upper triangular σ of the form

$$\underline{\sigma}(a) = \begin{pmatrix} \alpha(a) & \delta(a) \\ 0 & a \end{pmatrix}$$

$\delta : A \longrightarrow A$ is an α-derivation and we can consider both $R = A[t; \alpha, \delta]$ and $B = A[\underline{t}; \underline{\sigma}]$ where $\underline{t} = \binom{t_1}{t_2}$. Let us remark that the map $\varphi : B \to R$ defined by $\varphi\,(t_1) = t, \varphi\,(t_2) = 1$ and $\varphi(a) = a$ for all $a \in A$ is a ring homomorphism between B and R.

4) Let us briefly mention that we can extend the procedure developed in (3) above and consider a general map $\underline{\sigma} : A \to UT_n(A)$, where $UT_n(A)$ stands for upper triangular matrices. There will be rings maps from Ore extensions with n variables to rings construct with less than n variables coming form the same matrix of maps.

5) Another point of view on the above construction is obtained by remarking that it is corresponds to a filtered ring freely and finitely generated (n generators) in degree 1.

Definition 3.11. *Let V be a left A-module and $S_1, \ldots, S_n \in \operatorname{End}(V,+)$ be such that, for $a \in A$ and $v \in V$, we define*

$$S_i(av) = \sum_{j=1}^{n} \sigma_{ij}(a) S_j(v) + \delta_i(a) v. \quad \forall 1 \leq i \leq n.$$

A sequence of maps satisfying these equations will be called a ($\underline{\sigma}, \underline{\delta}$)-pseudo-multilinear transformation ($(\underline{\sigma}, \underline{\delta}) - PMT$, for short) on V.

Example 3.12. *Let $\underline{a} = (a_1, \ldots, a_n)^t$ be a column in A^n then the PMT on A defined as follows $S_{\underline{a}} = (S_{a_1}, \ldots, S_{a_n})^t$ with*

$$S_{a_i}(b) = \sum_{j=1}^{n} \sigma_{ij}(b) a_j + \delta_i(b)$$

We can check that we indeed get a PMT defined over A. As we will see, this PMT is closely related to the evaluation at $\underline{a}$.

Proposition 3.13. *Let S be a PMT defined on a left B-module V. Then*

(1) The following map

$$\varphi : B \to \operatorname{End}(V,+) \text{ such that } \varphi(f(\underline{t})) = f(\underline{S}),$$

is a ring homomorphism.

(2) There is a 1-1 correspondence between the set of PMT's defined on V and the set of B-module structures on V.

Definition 3.14. *We define the evaluation of $f(\underline{t}) \in B = A[\underline{t}; \sigma, \underline{\delta}]$ at $(a_1, \ldots, a_n) \in A^n$, via the representative of $f(\underline{t}) + I \in B/I$ by an element of A, where I is the left ideal $I = B(t_1 - a_1) + B(t_2 - a_2) + \cdots + B(t_n - a_n)$.*

2. If $x \in U(A)$ we denote $\underline{a}^x$ the ($\sigma, \underline{\delta}$)-conjugate of $\underline{a}$ (a column in A^n) by x defined by

$$\underline{a}^x = \sigma(x) \underline{a} x^{-1} + \underline{\delta}(x) x^{-1}$$

3. For $\underline{a}, \underline{b} \in A^n$ we define $\underline{a} \sim \underline{b}$ if there exists a nonzero divisor $x \in A$ such that $\underline{b} x = \sigma(x) \underline{a} + \underline{\delta}(x)$. We put

$$\Delta(\underline{a}) = \{\underline{b} \in A^n \mid \underline{a} \sim \underline{b}\}.$$

This definition was introduced by Martinez-Peñas in [29] and used in the context of coding theory.

Proposition 3.15. *For $f(\underline{t}) \in B = A[\underline{t}, \underline{\sigma}, \underline{\delta}]$ and $\underline{a} \in A^n$ we have $f(\underline{a}) = f\left(T_{\underline{a}}\right)(1)$.*

Proposition 3.16. *Suppose that $f, g \in B, \underline{a} \in A^n$, and $x \in A$.*

(1)

$$(fg)(\underline{a}) = f\left(S_{\underline{a}}\right)(g(\underline{a}))$$

In particular, if $g(\underline{t}) = x \in A$*, then we have* $(fx)(\underline{a}) = f\left(S_{\underline{a}}\right)(x)$*.*

(2) Assume that $0 \neq g(a) \in U(A)$*, then we get:*

$$(fg)(\underline{a}) = f\left(\underline{a}^{g(\underline{a})}\right) g(\underline{a}).$$

In particular, if $g(\underline{t}) = x \in A$*, then we have* $(f \circ x)(\underline{a}) = f\left(S_{\underline{a}}\right)(x)$*.*

(3) Assume that $0 \neq g(a) \in U(A)$*, then we get:*

$$(fg)(\underline{a}) = f\left(\underline{a}^{g(\underline{a})}\right) g(\underline{a}).$$

c) **An Application, Werner's question.**

In the submitted work [1] the authors solved and generalized a question raised by Werner [36]. Let A be a finite ring and consider the set

$$K(A) =: \{f(x) \in A[x] \mid f(a) = 0, \forall a \in A\}.$$

This set is easily seen to be a left ideal of $A[x]$. The question:

"Is $K(A)$ a 2-sided ideal?"

In a recent work this conjecture was solved in a more general setting.

Theorem 3.17. *Let A be a good ring (every element is a sum of units) and let $\sigma \in End(R)$ and δ a σ-derivation. Then the set $\{f(t) \in R = A[t;\sigma,\delta] \mid f(A) = 0\}$ is a two sided ideal of R. In particular, this is true if A is finite.*

A more general assumption gives the result.

Definition 3.18. *An element $\alpha \in A$ is (σ, δ) right invariant if*

$$\forall a \in A,\ \exists b \in A : T_a(\alpha) = b\alpha$$

Remark 3.19. *(1) $\alpha \in A$ is $(id., 0)$-right invariant iff $\alpha R \subseteq R\alpha$.*

(2) Any unit of A is (σ, δ)-right invariant.

Lemma 3.20. *Let $\alpha \in A$ be (σ, δ)-right invariant then for any $f(t) \in A[t;\sigma,\delta]$ there exists $b \in A$ such that:*

$$f(T_a)(\alpha) = f(b)\alpha$$

Theorem 3.21. *Let A be a ring such that every element of A is a sum of (σ, δ)-right invariant elements. Then the set $K(A, \sigma, \delta)$ is a two-sided ideal.*

Proof. $f(t), g(t) \in A[t;\sigma,\delta]$ with $f(t) \in K(A, \sigma, \delta)$ and for $a \in A$,

$$(f(t)g(t))(a) = f(T_a)(g(a)).$$

There exist (σ, δ)-right invariant elements $u_1, \dots, u_s$ such that $g(r) = u_1 + \dots + u_s$. Using Lemma 3.20, we get $f(T_a)(u_i) = f(b_i)u_i = 0$, for some $b_i \in A$. This gives $f(T_a)(u_1 + \dots + u_s) = \sum f(T_a)(u_i) = 0$. □

With an appropriate notion of $(\underline{\sigma}, \underline{\delta})$-right invariant elements and using the pseudo multi linear maps introduced above, this result can be extended to multi-variables polynomials $B = A[\underline{t}; \underline{\sigma}, \underline{\delta}]$.

Open questions

There are many open questions. Let us end this survey by mentioning some of them.

1) About Werner's question, it turns out that the actual difficult question would rather be to have an example of a killing ideal that is not two-sided.

2) What could be a good generalization of Cayley Hamilton theorem for PLT.

3) To what extent can we develop algebraic geometry for skew polynomials? The case of Ore polynomials with a single variable and with coefficients over a division ring is meaningful and has been treated in Lam-Leroy's papers. The case of multi variables over a division was considered in [29]. For some results in this direction the reader can also refer to [30].

4) Is it possible to develop a Galois theory for PLT? There exists a Galois theory for extensions of division rings, developed by Cartan [5]. The differential Galois theory is well developed [15], [33].

Acknowledgement

The Author would like to thank the referee for his careful reading and his useful comments.

References

[1] M. Aryapoor, A. Leroy, H. Merdach, *Annihilating polynomials*, submitted.

[2] D. Boucher, W. Geiselmann and F. Ulmer, *Skew cyclic codes*, Appl. Algebra Engrg. Comm. Comput. **18** (2007), 379–389.

[3] D. Boucher and F. Ulmer, *Coding with skew polynomial rings*, J. Symbolic Comput. **44** (2009), 1644–1656.

[4] M. Boulagouaz, A. Leroy, *(σ, δ)-codes*, Adv. Math. Commun. **7** (2013), no. 4, 463–474.

[5] H. Cartan, *Théorie de Galois pour les corps non commutatifs*, Ann. Sci. École Norm. Sup. (3) **64** (1947), 59–77.

[6] P. M. Cohn, *Free Rings and Their Relations*, London Math. Soc. Monogr., Academic Press, London–New York, 1971.

[7] P. M. Cohn, *Skew Fields, Theory of General Division Rings*, Encyclopaedia Math. Sci. **57**, Cambridge Univ. Press, Cambridge, 1995.

[8] J. Dauns, *A Concrete Approach to Division Rings*, Heldermann, Berlin, 1982.

[9] P. Fuchs, C. Maxson, G. Pilz, *Rings with FZP*, Trans. Amer. Math. Soc. **349** (1997), no. 3, 1271–1284.

[10] B. Gordon, T. S. Motzkin, *On the zeros of polynomials over division rings*, Trans. Amer. Math. Soc. **116** (1965), 218–226.

[11] N. Jacobson, *Pseudo-linear transformations*, Ann. of Math. (2) **38** (1937), 484–507.

[12] N. Jacobson, *The Theory of Rings*, Math. Surveys, No. 2, Amer. Math. Soc., Providence, RI, 1943.

[13] N. Jacobson, *The theory of rings*, Amer. Math. Soc., Providence, RI, 1943.

[14] N. Jacobson, *Finite-Dimensional Division Algebras over Fields*, Springer, Berlin, 1996.

[15] E. R. Kolchin, *Galois theory of differential fields*, Amer. J. Math. **75** (1953), 753–824.

[16] E. Kreindler, *Opérateurs additifs dans les modules unitaires à base dénombrable*, Bull. Math. Soc. Sci. Math. Roumanie **9** (1965), no. 4, 333–336.

[17] A. Leroy, *Pseudo-linear transformations and evaluation in Ore extensions*, Bull. Belg. Math. Soc. **2** (1995), 321–347.

[18] A. Leroy, *Noncommutative polynomial maps*, J. Algebra Appl. **11** (2012), no. 4.

[19] A. Leroy, J. Matczuk, *The extended centroid and X-inner automorphisms of Ore extensions*, J. Algebra **145** (1992), no. 1, 143–177.

[20] A. Leroy, H. Merdach, *Polynomial evaluations in noncommutative settings, a survey*, to appear in J. Algebra Appl. (2026).

[21] A. Leroy, A. Ozturk, *Algebraic and F-independent sets in* 2*-firs*, Comm. Algebra **32** (2004), no. 5, 1763–1792.

[22] T. Y. Lam and A. Leroy, *Vandermonde and Wronskian matrices over division rings*, J. Algebra **119** (1988), 308–336.

[23] T. Y. Lam and A. Leroy, *Algebraic conjugacy classes and skew polynomial rings*, in "Perspectives in Ring Theory", Kluwer Acad. Publ., Dordrecht, 1988, pp. 153–203.

[24] T. Y. Lam and A. Leroy, *Hilbert 90 theorems for division rings*, Trans. Amer. Math. Soc. **345** (1994), 595–622.

[25] T. Y. Lam and A. Leroy, *Wedderburn polynomials over division rings, I*, J. Pure Appl. Algebra **186** (2004), 43–76.

[26] T. Y. Lam, A. Leroy, A. Ozturk, *Wedderburn polynomials over division rings, II*, Contemp. Math. **456** (2008), 73–98.

[27] T. Y. Lam, A. Leroy, K. H. Leung, J. Matczuk, *Invariant and semi-invariant polynomials*, Israel Math. Conf. Proc. **1** (1989), 247–261.

[28] B. Marinescu, H. Bourlès, *Linear Time-Varying Systems Algebraic-Analytic Approach*, Lecture Notes Control Inf. Sci. **410**, Springer, 2011.

[29] U. Martínez-Peñas and F. R. Kschischang, *Evaluation and interpolation over multivariate skew polynomial rings*, J. Algebra **525** (2019), 111–139.

[30] C. Millet, *Pseudo-linear algebra over a division ring*, J. Algebra **569** (2021), 143–168.

[31] Ø. Ore, *Theory of noncommutative polynomials*, Ann. of Math. (2) **34** (1933), 480–508.

[32] J. Öinert, J. Richter, S. D. Silvestrov, *Maximal commutative subrings and simplicity of Ore extensions*, J. Algebra Appl. **12** (2013), no. 4, 1250192.

[33] M. van der Put, M. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren Math. Wiss. **328**, Springer, Berlin, 2003.

[34] L. H. Rowen, *Polynomials over division rings, and their applications*, World Sci. Publ., Singapore, 1993, pp. 287–301.

[35] J. H. M. Wedderburn, *On division algebras*, Trans. Amer. Math. Soc. **22** (1921), 129–135.

[36] N. J. Werner, *Polynomials that kill each element of a finite ring*, J. Algebra Appl. **13** (2014), no. 3, 1350111.