



## On the condition of the order of a periodic derivation in low dimensional complex Leibniz algebras

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### ABSTRACT

This study aims to investigate the low dimensional complex Leibniz algebras which admit a periodic derivation. The principal goal of this note is to characterize such algebras and to develop some properties on periodic derivations. We demonstrate that finite dimensional complex Leibniz algebras admitting a periodic derivation are abelian or at most 2-class nilpotent. Moreover, we prove that the order of a periodic derivation in such algebras is divided by 6.

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## 1 Introduction and preliminaries

Loday [8] in 1993 described Leibniz algebra which is a generalization of Lie algebra. Jacobson [6] has demonstrated that a Lie algebra with characteristic zero satisfying a non-singular derivation is nilpotent. Conversely, Dixmier and Lister [5] gave an example to prove that each nilpotent Lie algebra does not admit a non-singular derivation. Since a periodic derivation is non-singular, Burde [3] obtained that each Lie algebra satisfying a periodic derivation is nilpotent. By using the explicit calculations, it is seen that some

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2-class nilpotent Lie algebras satisfy a periodic derivation. But it does not hold in general, this is, every 2-class nilpotent Lie algebra does not admit a periodic derivation. There exist counterexamples in dimension 7. By using the same methods in [6], the authors in [4] showed that a Leibniz algebra which has a non-singular derivation is nilpotent. Furthermore, a Leibniz algebra admitting an automorphism of prime period which has no non-zero fixed points is nilpotent. Our primary beginning motivation is presented by the work [3] of Burde in 2012 which started the study of complex Lie algebras having a periodic derivation. The principal goal of this note is to characterize low dimensional non-Lie complex Leibniz algebras having a periodic derivation and to develop some properties of such algebras. Then we show that a Leibniz algebra which admits a periodic derivation is nilpotent but every nilpotent Leibniz does not have a periodic derivation. We construct some examples. In summary, we have shown that complex Leibniz algebras admitting a periodic derivation are abelian or nilpotent of at most class 2, and further it is proved that the order of periodic derivation in these algebras is divided by 6.

Throughout this part, some definitions and properties are introduced by using references [1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15]. First, we describe some observations about derivations. Let  $(\mathcal{A}, \cdot)$  be an algebra over a field  $\mathcal{K}$  with characteristic zero and  $\delta$  be its a linear mapping. If this mapping satisfies  $\delta(\vartheta.\gamma) = \delta(\vartheta).\gamma + \vartheta.\delta(\gamma)$  for every  $\vartheta, \gamma \in \mathcal{A}$ , it is said to be a derivation of  $\mathcal{A}$ . By  $der(\mathcal{A})$ , we represent the set of all derivations of  $\mathcal{A}$ . The definitions of left and right multiplication operators  $l_\kappa, r_\kappa$  are described by  $l_\kappa(\gamma) = \kappa.\gamma$  and  $r_\kappa(\gamma) = \gamma.\kappa$  for  $\gamma \in \mathcal{A}$ , respectively. Assume that  $\mathcal{A}$  is an  $m$ -dimensional algebra and  $\delta$  is its derivation. Take a basis  $\mathcal{B} = \{\kappa_1, \kappa_2, \dots, \kappa_m\}$  of  $\mathcal{A}$ . Depending on this basis,  $\delta$  is denoted by a matrix  $[\delta]_{\mathcal{B}} = [\delta_{ij}]$ . By solving the system of equations

$$\delta \circ r_{\kappa_i} = r_{\delta(\kappa_i)} + r_{\kappa_i} \circ \delta \iff \delta \circ r_{\kappa_i} - r_{\kappa_i} \circ \delta = r_{\delta(\kappa_i)} \quad (1)$$

for all  $i = 1, 2, \dots, m$ , we get the entries of the matrix  $[\delta]_{\mathcal{B}}$ , so that we find the matrix  $[\delta]_{\mathcal{B}} = [\delta_{ij}]$ . Moreover, the derivation  $\delta$  is said to be semi-simple if  $\delta(\kappa_i) = \mathcal{K}\kappa_i$ , for some linear basis  $\{\kappa_1, \kappa_2, \dots, \kappa_m\}$ .

If an algebra  $\mathcal{S}$  over  $\mathcal{K}$  with the operation  $[\cdot, \cdot] : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  holds the Leibniz identity

$$[\kappa, [\beta, \gamma]] = [[\kappa, \beta], \gamma] - [[\kappa, \gamma], \beta] \quad (2)$$

for every  $\kappa, \beta, \gamma \in \mathcal{S}$ , then we say that  $\mathcal{S}$  is a (right) Leibniz algebra. The right Leibniz algebra is always used throughout this work. Recall that Leibniz algebra is a non-anticommutative generalization of Lie algebra. As a prompt result, each Lie algebra is a Leibniz algebra. Given a subspace  $\mathcal{T}$  of a Leibniz algebra  $\mathcal{S}$ , it is said that  $\mathcal{T}$  is a sub-algebra if  $[\kappa, \gamma] \in \mathcal{T}$  for each  $\kappa, \gamma \in \mathcal{T}$ . Moreover, if  $[\kappa, \gamma] \in \mathcal{T}$  and  $[\gamma, \kappa] \in \mathcal{T}$  for every  $\kappa \in \mathcal{S}$  and  $\gamma \in \mathcal{T}$ , then we say that  $\mathcal{T}$  is an ideal of  $\mathcal{S}$ . By  $Leib(\mathcal{S})$ , we denote the ideal of  $\mathcal{S}$  generated by the elements  $[\kappa, \kappa], \kappa \in \mathcal{S}$ . This ideal is called the Leibniz kernel of the algebra  $\mathcal{S}$ .

The definition of the lower central series of  $\mathcal{S}$  is given by the series of ideals

$$\mathcal{S} = \mathcal{S}^1 \supseteq \mathcal{S}^2 \supseteq \dots \supseteq \mathcal{S}^k \supseteq \mathcal{S}^{k+1} \supseteq \dots$$

where for positive integer  $n$ ,  $\mathcal{S}^{n+1} = [\mathcal{S}, \mathcal{S}^n]$ . Further, we say nilpotent of class  $c$  if a Leibniz algebra holds that  $\mathcal{S}^{c+1} = 0$  but  $\mathcal{S}^c \neq 0$ . The series of ideals

$$\mathcal{S} = \mathcal{S}^{(1)} \supseteq \mathcal{S}^{(2)} \supseteq \dots \supseteq \mathcal{S}^{(k)} \supseteq \mathcal{S}^{(k+1)} \supseteq \dots$$

where  $\mathcal{S}^{(n+1)} = [\mathcal{S}^{(n)}, \mathcal{S}^{(n)}]$  for positive integer  $n$ , is said to be the derived series of  $\mathcal{S}$ . We say a solvable Leibniz algebra  $\mathcal{S}$  if  $\mathcal{S}^{(n)} = 0$  for some integer  $n \geq 0$ . Each nilpotent Leibniz algebra is solvable. Recall that each non-Lie Leibniz algebra with  $\dim(\mathcal{S}) \leq 4$  is solvable ([4], Theorem 6.2).

## 2 Main results

### 2.1 Derivations of Leibniz algebras

A  $\mathcal{K}$ -linear mapping  $\delta$  of the Leibniz algebra  $\mathcal{S}$  is said to be a derivation of  $\mathcal{S}$  if  $\delta([\kappa, \gamma]) = [\delta(\kappa), \gamma] + [\kappa, \delta(\gamma)]$  for every  $\kappa, \gamma \in \mathcal{S}$ . The set consisting of all derivations in  $\mathcal{S}$  is represented by  $\text{der}(\mathcal{S})$ . Let  $\kappa \in \mathcal{S}$ . The left multiplication operator  $l_\kappa$  and the right multiplication operator  $r_\kappa$  are defined as  $l_\kappa(\gamma) = [\kappa, \gamma]$  and  $r_\kappa(\gamma) = [\gamma, \kappa]$  for  $\gamma \in \mathcal{S}$ , respectively. By using (2), it is shown that  $r_\kappa$  is a derivation, but  $l_\kappa$  is not a derivation. Indeed,

$$\begin{aligned} r_\kappa([\vartheta, \gamma]) &= [[\vartheta, \gamma], \kappa] \\ &= [\vartheta, [\gamma, \kappa]] + [[\vartheta, \kappa], \gamma] \\ &= [\vartheta, r_\kappa(\gamma)] + [r_\kappa(\vartheta), \gamma] \end{aligned}$$

for  $\vartheta, \gamma \in \mathcal{S}$ . Moreover, the space  $R_\mathcal{S} = \{r_\kappa \mid \kappa \in \mathcal{S}\}$  is a Lie algebra with the commutator product.

Let  $\mathcal{S}$  be one dimensional Leibniz algebra, this is,  $\mathcal{S} = \mathcal{K}\alpha$  for some element  $\alpha \in \mathcal{S}$ . Then  $[\alpha, \alpha] = \lambda\alpha$  where  $\lambda \in \mathcal{K}$ . We have

$$0 = [[\alpha, \alpha], \alpha] = [\lambda\alpha, \alpha] = \lambda[\alpha, \alpha] = \lambda^2\alpha.$$

Therefore  $\lambda = 0$ , that is,  $[\alpha, \alpha] = 0$  and  $\mathcal{S}$  is abelian.

**Lemma 2.1.** *Let  $\mathcal{S}$  be a two dimensional Leibniz algebra. Then  $\mathcal{S}$  is isomorphic to a Leibniz algebra given by one of the following*

- (i) abelian,
- (ii)  $[\vartheta_1, \vartheta_2] = -[\vartheta_2, \vartheta_1] = \vartheta_2$ ,
- (iii)  $[\vartheta_1, \vartheta_2] = \vartheta_1, [\vartheta_2, \vartheta_2] = \vartheta_1$ ,
- (iv)  $[\vartheta_2, \vartheta_2] = \vartheta_1$

where  $\{\vartheta_1, \vartheta_2\}$  is a basis of  $\mathcal{S}$ .

By using (1) we find the derivation algebras and the basis derivations of the derivation algebras.

**Example 2.2.** Let  $\mathcal{S}$  be two dimensional Leibniz algebra with the multiplication table  $[\alpha_1, \alpha_2] = \alpha_1, [\alpha_2, \alpha_2] = \alpha_1$  on a basis  $\mathcal{B} = \{\alpha_1, \alpha_2\}$ . Then  $\{\delta_1, \delta_2\}$  with  $\delta_1(\alpha_1) = \alpha_1$  and  $\delta_2(\alpha_2) = \alpha_1$  forms a basis of the algebra  $\text{der}(\mathcal{S})$ . Let  $\delta \in \text{der}(\mathcal{S})$ ,

$$[\delta]_{\mathcal{B}} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} \text{ and } [r_{\alpha_1}]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, [r_{\alpha_2}]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

By (1), we have  $[\delta]_{\mathcal{B}} \circ [r_{\alpha_2}]_{\mathcal{B}} - [r_{\alpha_2}]_{\mathcal{B}} \circ [\delta]_{\mathcal{B}} = \delta_{11}[r_{\alpha_1}]_{\mathcal{B}} + \delta_{21}[r_{\alpha_2}]_{\mathcal{B}}$ . Then we get

$$\begin{aligned} \begin{pmatrix} \delta_{11} & \delta_{11} \\ \delta_{21} & \delta_{22} \end{pmatrix} - \begin{pmatrix} \delta_{11} + \delta_{21} & \delta_{21} + \delta_{22} \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -\delta_{21} & \delta_{11} - \delta_{12} - \delta_{22} \\ \delta_{21} & \delta_{22} \end{pmatrix} \\ &= \begin{pmatrix} \delta_{21} & \delta_{21} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and so that

$$\delta_{21} = 0, \delta_{22} = 0, \delta_{11} = \delta_{12}. \quad (3)$$

From (3), we obtain

$$[\delta]_{\mathcal{B}} = \begin{pmatrix} \delta_{11} & \delta_{11} \\ 0 & 0 \end{pmatrix} = \delta_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \delta_{11} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the basis of  $\text{der}(\mathcal{S})$  is  $\{\delta_1, \delta_2\}$ , where  $\delta_1(\alpha_1) = \alpha_1$  and  $\delta_2(\alpha_2) = \alpha_1$ .

In dimension two, there are three complex Leibniz algebras and their dimensions are one or two.

Table 1: Derivations of 2-dimensional Leibniz algebras

<u>multiplication table</u>	<u>basis derivations</u>
$\mathcal{S}_1: [\alpha_1, \alpha_2] = -[\alpha_2, \alpha_1] = \alpha_2$	$\delta_1(\alpha_1) = \alpha_2, \delta_2(\alpha_2) = \alpha_2$
$\mathcal{S}_2: [\alpha_2, \alpha_2] = \alpha_1, [\alpha_1, \alpha_2] = \alpha_1$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \alpha_1$
$\mathcal{S}_3: [\alpha_2, \alpha_2] = \alpha_1$	$\delta_1(\alpha_i) = \frac{1}{i}\alpha_i, 1 \leq i \leq 2, \delta_2(\alpha_2) = \alpha_1$

In Table 1, we observe that  $\mathcal{S}_1$  is a non-nilpotent and solvable complex Leibniz algebra and  $\mathcal{S}_2$  and  $\mathcal{S}_3$  are nilpotent of class two. Now we observe derivations of three dimensional complex Leibniz algebras in Table 2. There exist fourteen three dimensional complex Leibniz algebras and the dimension of derivation algebra ranges between one and four.

## 2.2 Periodic derivations

Now we present the description of periodic derivation and its properties in this part.

**Definition 2.3.** Let  $\mathcal{S}$  be a Leibniz algebra and  $\delta$  be its derivation. This derivation  $\delta$  is said to be periodic if there is a positive integer  $n$  satisfying  $\delta^n = \text{id}$ .

Table 2: Derivations of 3-dimensional Leibniz algebras

<u>multiplication table</u>	<u>basis derivations</u>
$\mathcal{S}_1: [\alpha_3, \alpha_3] = \alpha_1, [\alpha_1, \alpha_3] = \alpha_1 + \alpha_2$	$\delta_1(\alpha_1) = \alpha_1 + \alpha_2, \delta_1(\alpha_3) = \alpha_1, \delta_2(\alpha_3) = \alpha_2$
$\mathcal{S}_2: [\alpha_3, \alpha_3] = \alpha_1, [\alpha_1, \alpha_3] = \lambda\alpha_1$	$\delta_1(\alpha_1) = \lambda\alpha_1, \delta_1(\alpha_3) = \alpha_1$
$[\alpha_3, \alpha_2] = \alpha_2[\alpha_2, \alpha_3] = -\alpha_2$	$\delta_2(\alpha_2) = \alpha_2, \delta_3(\alpha_3) = \alpha_2$
$\mathcal{S}_3: [\alpha_3, \alpha_3] = \lambda\alpha_1, \lambda \neq 0$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \alpha_3, \delta_1(\alpha_3) = -\lambda\alpha_2 + \alpha_3$
$[\alpha_2, \alpha_3] = \alpha_1, [\alpha_2, \alpha_2] = \alpha_1$	$\delta_2(\alpha_2) = \alpha_1, \delta_3(\alpha_2) = \alpha_2 - 2\alpha_3,$
	$\delta_3(\alpha_3) = 2\lambda\alpha_2 - \alpha_3, \delta_4(\alpha_3) = \alpha_1$
$\mathcal{S}_4: [\alpha_2, \alpha_3] = \lambda\alpha_1 + \alpha_2, \lambda \neq 0, [\alpha_1, \alpha_3] = \alpha_2$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \alpha_2,$
	$\delta_2(\alpha_1) = \alpha_2, \delta_2(\alpha_2) = \lambda\alpha_1 + \alpha_2$
$\mathcal{S}_5: [\alpha_1, \alpha_3] = \alpha_2, [\alpha_2, \alpha_3] = \alpha_1$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \alpha_2,$
	$\delta_2(\alpha_1) = \alpha_2, \delta_2(\alpha_2) = \alpha_1$
$\mathcal{S}_6: [\alpha_3, \alpha_3] = \alpha_1, [\alpha_2, \alpha_2] = \alpha_1$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \frac{1}{2}\alpha_2, \delta_1(\alpha_3) = \frac{1}{2}\alpha_3$
	$\delta_2(\alpha_2) = \alpha_1, \delta_3(\alpha_2) = \alpha_3,$
	$\delta_3(\alpha_3) = -\alpha_2, \delta_4(\alpha_3) = \alpha_1$
$\mathcal{S}_7: [\alpha_1, \alpha_3] = \alpha_1, [\alpha_2, \alpha_3] = \alpha_2$	$\delta_1(\alpha_1) = \alpha_1, \delta_2(\alpha_1) = \alpha_2,$
	$\delta_3(\alpha_2) = \alpha_1, \delta_4(\alpha_2) = \alpha_2$
$\mathcal{S}_8: [\alpha_3, \alpha_3] = \alpha_1, [\alpha_1, \alpha_3] = \alpha_2$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \frac{3}{2}\alpha_2, \delta_1(\alpha_3) = \frac{1}{2}\alpha_3,$
	$\delta_2(\alpha_1) = \alpha_2, \delta_2(\alpha_3) = \alpha_1, \delta_3(\alpha_3) = \alpha_2$
$\mathcal{S}_9: [\alpha_1, \alpha_3] = -2\alpha_1, [\alpha_2, \alpha_2] = \alpha_1,$	$\delta_1(\alpha_i) = \frac{1}{i}\alpha_i, 1 \leq i \leq 2,$
$[\alpha_2, \alpha_3] = -\alpha_2, [\alpha_3, \alpha_2] = \alpha_2$	$\delta_2(\alpha_i) = \alpha_{i-1}, 2 \leq i \leq 3$
$\mathcal{S}_{10}: [\alpha_1, \alpha_2] = \alpha_1$	$\delta_1(\alpha_1) = \alpha_1, \delta_2(\alpha_2) = \alpha_3, \delta_3(\alpha_3) = \alpha_3$
$\mathcal{S}_{11}: [\alpha_1, \alpha_3] = \alpha_1 + \alpha_2, [\alpha_2, \alpha_3] = \alpha_2$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = \alpha_2, \delta_2(\alpha_1) = \alpha_2$
$\mathcal{S}_{12}: [\alpha_2, \alpha_3] = -\alpha_2, [\alpha_1, \alpha_3] = \alpha_1, [\alpha_1, \alpha_2] = \alpha_3$	$\delta_1(\alpha_1) = \alpha_1, \delta_1(\alpha_2) = -\alpha_2$
$\mathcal{S}_{13}: [\alpha_1, \alpha_3] = \alpha_1, [\alpha_2, \alpha_3] = \lambda\alpha_2, \lambda \neq 0, 1$	$\delta_1(\alpha_1) = \alpha_1, \delta_2(\alpha_1) = \alpha_2,$
	$\delta_3(\alpha_2) = \alpha_1, \delta_4(\alpha_2) = \alpha_2$
$\mathcal{S}_{14}: [\alpha_1, \alpha_3] = \alpha_1, [\alpha_2, \alpha_3] = \alpha_2$	$\delta_1(\alpha_1) = \alpha_1, \delta_2(\alpha_2) = \alpha_2$

**Remark 2.4.** Let  $\mathcal{S}$  be a Leibniz algebra satisfying a periodic derivation  $\delta$  with  $n = 1$ , then we obtain

$$\begin{aligned}
[\kappa, \epsilon] &= \delta([\kappa, \epsilon]) \\
&= [\delta(\kappa), \epsilon] + [\kappa, \delta(\epsilon)] \\
&= [\kappa, \epsilon] + [\kappa, \epsilon] \\
&= 2[\kappa, \epsilon] \\
0 &= [\kappa, \epsilon],
\end{aligned}$$

it follows that  $\mathcal{S}$  is abelian. Contrarily, an abelian Leibniz algebra has periodic derivations of any possible integer order  $n$ . In fact, only we take  $\delta = \varsigma id \in \text{der}(\mathcal{S})$ , where  $\varsigma$  is a primitive  $n$ -th root of unity.

**Remark 2.5.** Let  $\mathcal{S}$  be a Leibniz algebra. If  $\text{Leib}(\mathcal{S}) \neq \{0\}$ , then this algebra admits no periodic derivation. Clearly, assume that  $\delta$  is a periodic derivation of a positive integer  $n$  of  $\mathcal{S}$  and  $[\kappa, \kappa] \in \text{Leib}(\mathcal{S})$ . Then since  $\delta$  is a periodic derivation of  $\mathcal{S}$ , it is semi-simple.

Suppose that  $\delta(\kappa) = \lambda\kappa$ , so  $\delta([\kappa, \kappa]) = [\delta(\kappa), \kappa] + [\kappa, \delta(\kappa)] = 2\lambda[\kappa, \kappa]$ . By definition of periodic derivation, we have  $\lambda^n = 1$  and  $(2\lambda)^n = 1$ . But there does not exist a positive integer  $n$ . Therefore, this is a contradiction.

**Example 2.6.** Let  $\mathcal{S}$  be a two dimensional complex Leibniz algebra with multiplication  $[\alpha_2, \alpha_2] = \alpha_1$  as  $\mathcal{S}_3$  in Table 1. This algebra is nilpotent of class two and it has no periodic derivation.

We focus on non-abelian complex Leibniz algebras having a periodic derivation.

**Corollary 2.7** ([3], Corollary 2.3). Let  $s, t, p \in \mathbb{C}$ . Then at least one of the numbers  $s, t, p, s+p, t+p, s+t+p$  is not an  $n$ -th root of unity.

**Lemma 2.8** ([3], Lemma 2.2). Let  $s, t \in \mathbb{C}$  with  $|s| = |t| = |s+t| = 1$ . Then  $t = \varsigma s$  where  $\varsigma$  is a primitive 3rd root of unity.

**Corollary 2.9** ([4], Corollary 4.10). Let  $\mathcal{S}$  be a Leibniz algebra having a non-singular derivation. Then  $\mathcal{S}$  is nilpotent.

**Theorem 2.10.** Let  $\mathcal{S}$  be a Leibniz algebra which admits a periodic derivation. Then  $\mathcal{S}$  is nilpotent.

*Proof.* Suppose that  $\delta$  is a periodic derivation of  $\mathcal{S}$ . Since every periodic derivation is non-singular, by Corollary 2.9,  $\mathcal{S}$  becomes nilpotent.  $\square$

**Proposition 2.11.** Let  $\mathcal{S}$  be a complex Leibniz algebra having a periodic derivation. Then the nilpotent of class of  $\mathcal{S}$  is at most 2.

*Proof.* Suppose that  $\delta$  is a periodic derivation of  $\mathcal{S}$ . Therefore it is semi-simple. Assume that  $\mathcal{S}^3 = [\mathcal{S}, \mathcal{S}^2] \neq 0$ . Then there are eigenvectors  $\kappa, \vartheta, \epsilon$  of  $\delta$  with eigenvalues  $s, t, p$  such that  $[\epsilon, [\kappa, \vartheta]] \neq 0$ . Hence  $[\kappa, \vartheta] \neq 0$ . Therefore  $\delta([\kappa, \vartheta]) = [\delta(\kappa), \vartheta] + [\kappa, \delta(\vartheta)] = (s+t)[\kappa, \vartheta]$ . This means  $s^n = t^n = (s+t)^n = 1$ . By the Leibniz identity we may assume that also  $[[\epsilon, \kappa], \vartheta] \neq 0$ , hence  $[\epsilon, \kappa] \neq 0$ . Similarly,  $\delta([\epsilon, \kappa]) = [\delta(\epsilon), \kappa] + [\epsilon, \delta(\kappa)] = (s+p)[\epsilon, \kappa]$  and  $\delta([\vartheta, \epsilon]) = [\delta(\vartheta), \epsilon] + [\vartheta, \delta(\epsilon)] = (t+p)[\vartheta, \epsilon]$ . This means  $s^n = p^n = (s+p)^n = 1$  and  $t^n = p^n = (t+p)^n = 1$ . We accomplish that all of  $s, t, p, s+t, s+p, s+t+p$  are  $n$ -th roots of unity. This is in contradiction with Corollary 2.7.  $\square$

**Proposition 2.12.** Let  $\mathcal{S}$  be a complex Leibniz algebra having a periodic derivation  $\delta$  with  $\dim(\text{Leib}(\mathcal{S})) = 0$ . Then the inverse of  $\delta$  is again a derivation of  $\mathcal{S}$ .

*Proof.* Since  $\delta$  is a periodic derivation of  $\mathcal{S}$ , it is semi-simple. Assume that  $\{a_1, a_2, \dots, a_m\}$  is a basis of eigenvectors with  $\beta_1, \beta_2, \dots, \beta_m$ . For two non-commuting eigenvectors  $a_i$  and  $a_j$  with eigenvalues  $\beta_i$  and  $\beta_j$ , we have  $\beta_i = \varsigma\beta_j$  where  $\varsigma$  is a primitive 3rd root of unity. Furthermore, we get

$$\beta_i^{-1} + \beta_j^{-1} = \beta_j^{-1}(1 + \varsigma^{-1}) = \beta_j^{-1}(1 + \varsigma)^{-1} = (\beta_i + \beta_j)^{-1}. \quad (4)$$

By (4), we obtain  $\delta^{-1}([a_i, a_j]) = [\delta^{-1}(a_i), a_j] + [a_i, \delta^{-1}(a_j)]$ .  $\square$

**Theorem 2.13.** *Let  $\mathcal{S}$  be a non-abelian complex Leibniz algebra which has a periodic derivation  $\delta$  of order  $n$  with  $\dim(\text{Leib}(\mathcal{S})) = 0$ . Then  $6|n$ .*

*Proof.* Since  $\delta$  is a periodic derivation of  $\mathcal{S}$ , it is semi-simple and as  $\mathcal{S}$  is non-abelian, there are eigenvectors  $\kappa$  and  $\epsilon$  with eigenvalues  $s$  and  $t$  such that  $[\kappa, \epsilon]$  is a non-zero eigenvector with eigenvalue  $s + t$ . Indeed,

$$\delta([\kappa, \epsilon]) = [\delta(\kappa), \epsilon] + [\kappa, \delta(\epsilon)] = [s\kappa, \epsilon] + [\kappa, t\epsilon] = (s + t)[\kappa, \epsilon].$$

This means

$$s^n = t^n = (s + t)^n = 1,$$

so that  $t = \zeta s$  where  $\zeta$  is a primitive 3rd root of unity, by Lemma 2.8. Then we have  $s + t = s(1 + \zeta)$ . By raising this to the  $n$ -th power we conclude  $(1 + \zeta)^n = 1$  where  $(1 + \zeta)$  is a primitive 6-th root of unity, thus  $6|n$ .  $\square$

**Example 2.14.** *Let  $\mathcal{S}$  be a four dimensional complex Leibniz algebra with multiplications  $[\alpha_1, \alpha_2] = \alpha_3$ ,  $[\alpha_2, \alpha_1] = \alpha_4$  on a basis  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ . This algebra has a periodic derivation of order six defined by  $\delta(\alpha_1) = \zeta\alpha_1$ ,  $\delta(\alpha_2) = \alpha_2$ ,  $\delta(\alpha_3) = (\zeta + 1)\alpha_3$ ,  $\delta(\alpha_4) = (\zeta + 1)\alpha_4$ , where  $\zeta$  is a primitive 3rd root of unity. It is obvious to prove that  $\delta^{-1}$  is again derivation. Moreover, this algebra is 2-class nilpotent.*

There is no non-abelian one dimensional complex Leibniz algebra. In general, not every 2-class nilpotent complex Leibniz algebra has a periodic derivation. There are some counter examples in dimension two.

**Example 2.15.** *Let  $\mathcal{S}$  be a three dimensional complex Leibniz algebra with multiplications  $[\alpha_1, \alpha_3] = \alpha_2$ ,  $[\alpha_2, \alpha_3] = \lambda\alpha_1 + \alpha_2$ , ( $\lambda \neq 0$ ) as  $\mathcal{S}_4$  in Table 1. This algebra is 2-class nilpotent, but it has no periodic derivation.*

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