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Classification, derivations and centroids of low-dimensional associative trialgebras

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ABSTRACT

In this paper, we study the structure and algebraic varieties of associative trialgebras. In particular, we classify all associative trialgebras of dimension at most four over a field of characteristic zero. Based on this classification, we provide a detailed analysis of their derivations and centroids. We also investigate the role of centroids in the structural theory of associative trialgebras and compute them explicitly for each isomorphism class in low dimensions. All computations are performed using symbolic computation software such as `Mathematica`. These results offer new insights into the algebraic and geometric aspects of associative trialgebras.

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1 Introduction

Associative trialgebras, also known as triassociative algebras and denoted as $(\mathcal{T}, \perp, \dashv, \vdash)$, were first introduced by Loday and Ronco in 2001 (see [9]). These algebras extend the

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scope of Loday's associative dialgebras (diassociative algebras), as explored in the foundational works [2, 8, 10, 20]. Characterized by a vector space and three binary operations satisfying eleven defining relations, associative trialgebras have emerged as a rich field of study in the broader landscape of non-associative algebraic structures.

The classification of algebraic structures [1, 4, 17], including dialgebras and trialgebras [8, 20, 21], as well as Hom-type and BiHom-type generalizations [11, 13, 14, 22], has become a central focus in recent mathematical research. Derivations and centroids serve as crucial tools in the structural study of these algebras, offering deep insights into their internal symmetries and automorphism behaviors [6, 7, 15, 18]. These concepts not only contribute to pure algebra but also find applications in geometry and physics.

Recent developments in generalized and Hom-type algebraic frameworks have introduced new avenues for studying algebraic deformations, such as Hom-associative and Hom-trialgebra structures [3, 11, 12, 16], and their corresponding derivations and centroids [2, 13, 22, 23]. The role of Rota-Baxter relations, bimodule constructions, and Hom-Poisson structures is further emphasized in [2, 3, 12], highlighting the interplay between algebraic operations and homomorphisms in modern generalizations.

Let $\mathcal{T}$ be an n-dimensional $\mathbb{K}$-linear space with a basis $\{e_1, e_2, \ldots, e_n\}$. The triassociative structure on $\mathcal{T}$, characterized by the product operations γ, δ, and ξ, is governed by $3n^3$ structure constants $\gamma_{ij}^k, \delta_{ij}^k$, and ξ_{ij}^k. This structure is defined through the equations $e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k$, $e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^k e_k$, and $e_i \perp e_j = \sum_{k=1}^{n} \xi_{ij}^k e_k$. Ensuring the triassociative and unital properties gives rise to the sub-variety $\mathcal{T}_t$ of $\mathbb{K}^{3n^3}$.

Changes in $\mathcal{T}$ result in a natural transport of the structure action of $GL_n(\mathbb{K})$ on $\mathcal{T}_t$, establishing a one-to-one correspondence between isomorphism classes of n-dimensional algebras and the orbits of the action of $GL_n(\mathbb{K})$ on $\mathcal{T}_t$. The role of centroids in classification problems and various areas of algebraic structure theory is well-established [6, 7, 15, 18]. Similar approaches have been applied in the classification of low-dimensional Leibniz algebras and their derivations [19], as well as in Zinbiel algebras [4] and BiHom-superdialgebras [14].

This paper aims to introduce and classify derivations and centroids specifically within the context of associative trialgebras, building on earlier classifications of low-dimensional dialgebras [5, 8, 20] and utilizing computational techniques inspired by [1, 17, 16].

The paper is organized into several sections, each contributing to the understanding and classification of associative trialgebras. In the first section, we provide an introduction to the subject and highlight previously obtained results. Section 2 lays down the basic concepts essential for the ensuing study.

Section 3 delves into the algebraic varieties of associative trialgebras, offering classifications of two-dimensional, three-dimensional, and four-dimensional trialgebras up to isomorphism. The comprehensive analysis includes the revelation that any 2-dimensional associative trialgebra is isomorphic to one of 8 possible associative trialgebras. Similarly, we demonstrate that 3-dimensional associative trialgebras are isomorphic to one of the 12 possible non-isomorphic associative trialgebras, and 4-dimensional associative trialgebras are isomorphic to one of 16 possible associative trialgebras. Our meticulous classification involves solving structure constant equations with the aid of computer algebra software, as also adopted in works like [1, 16].

In Section 4, we present the classification of derivations, revealing 5 non-isomorphic derivations of two-dimensional associative trialgebras, 8 non-isomorphic derivations of three-dimensional associative trialgebras, and 16 non-isomorphic derivations of four-dimensional associative trialgebras, with dimensions ranging from 0 to 6. Our approach is influenced by methods used in earlier investigations of derivations in Hom-algebraic and dialgebraic settings [2, 3, 21, 22].

Finally, in Section 5, we delve into the classification of centroids. Our evaluation uncovers that the centroids of 2-dimensional associative trialgebras are isomorphic to each of the 8 non-isomorphic classes, each with a dimension of one. Similarly, the centroids of 3-dimensional associative trialgebras are isomorphic to each of the 12 non-isomorphic classes, with dimensions ranging from 1 to 5. Moreover, in the classification of 4-dimensional associative trialgebras and derivations, we identify 16 non-isomorphic centroids of associative trialgebras, with dimensions in the range of 1 to 7.

It is important to note that the concept of derivations and centroids in this context draws inspiration from that of finite-dimensional algebras [7, 18]. This study focuses specifically on the derivations and centroids of finite-dimensional associative trialgebras, showcasing their significance in algebraic and geometric classification problems of algebras. Utilizing the classification results of associative trialgebras, we provide a comprehensive description of derivations and centroids for two, three, and four-dimensional associative trialgebras. All algebras and vector spaces considered are assumed to be over a field $\mathbb{K}$ of characteristic zero. Our results significantly contribute to the understanding of the structure and properties of associative trialgebras, paving the way for future research in this intriguing area.

2 Preliminaries

Definition 2.1. *[9] An associative dialgebra is a vector space $\mathcal{D}$ equipped with two binary operations : $\dashv$ called left and $\vdash$ called right,*
(left) $\dashv: \mathcal{D}\times\mathcal{D}\to\mathcal{D}$ *and* *(right)* $\vdash: \mathcal{D}\times\mathcal{D}\to\mathcal{D}$ *satisfying the relations*

$$\begin{cases} (x\dashv y)\dashv z &= x\dashv(y\dashv z),\\ (x\dashv y)\dashv z &= x\dashv(y\vdash z),\\ (x\vdash y)\dashv z &= x\vdash(y\dashv z),\\ (x\dashv y)\vdash z &= x\vdash(y\vdash z),\\ (x\vdash y)\vdash z &= x\vdash(y\vdash z).\end{cases}$$

Definition 2.2. *[9] An associative trialgebra is a $\mathbb{K}$-vector space $(\mathcal{T},\perp,\dashv,\vdash)$ such that*

$(\mathcal{T}, \dashv, \vdash)$ *is an associative dialgebra,* $(\mathcal{T}, \perp)$ *an associative algebra and,*

$$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \dashv z), \\ (x \dashv y) \dashv z = x \dashv (y \vdash z), \\ (x \vdash y) \dashv z = x \vdash (y \dashv z), \\ (x \dashv y) \vdash z = x \vdash (y \vdash z), \\ (x \vdash y) \vdash z = x \vdash (y \vdash z), \end{cases}$$

$$\begin{cases} (x \dashv y) \dashv z = x \dashv (y \perp z), \\ (x \perp y) \dashv z = x \perp (y \dashv z), \\ (x \dashv y) \perp z = x \perp (y \vdash z), \\ (x \vdash y) \perp z = x \vdash (y \perp z), \\ (x \perp y) \vdash z = x \vdash (y \vdash z), \end{cases}$$

$$\{ \quad (x \perp y) \perp z = x \perp (y \perp z).$$

Definition 2.3. *Let* $(\mathcal{T}_1, \perp_1, \dashv_1, \vdash_1)$, $(\mathcal{T}_2, \perp_2, \dashv_2, \vdash_2)$ *be associative trialgebras over a field* $\mathbb{K}$*. Then a homomorphism from* $\mathcal{T}_1$ *to* $\mathcal{T}_2$ *is a* $\mathbb{K}$*-linear mapping* $\eta : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ *such that*

$$\eta(x \dashv_1 y) = \eta(x) \dashv_2 \eta(y) \tag{1}$$

$$\eta(x \vdash_1 y) = \eta(x) \vdash_2 \eta(y) \tag{2}$$

$$\eta(x \perp_1 y) = \eta(x) \perp_2 \eta(y) \tag{3}$$

for all $x, y \in \mathcal{T}_1$.

Remark 2.4. *A bijective homomorphism is an isomorphism of* $\mathcal{T}_1$ *and* $\mathcal{T}_2$.

Proposition 2.5. *Let* $(\mathcal{T}, \dashv, \perp, \vdash)$ *be an associative trialgebras. Then* $(\mathcal{T}, \star)$ *is an associative algebra. with respect to the multiplication* $* : \mathcal{T} \otimes \mathcal{T} \longrightarrow \mathcal{T}$:

$$x * y = x \dashv y + x \vdash y - x \perp y$$

for any $x, y \in \mathcal{T}$.

Proof. Using the axioms of associative trialgebra we have for $x, y, z \in \mathcal{T}$

$$
\begin{aligned}
(x*y)*z &= (x \dashv y + x \vdash y - x\perp y)*z \\
&= (x \dashv y + x \vdash y - x\perp y) \dashv z + (x \dashv y + x \vdash y - x\perp y) \vdash z \\
&\quad -(x \dashv y + x \vdash y - x\perp y)\perp z \\
&= (x \dashv y) \dashv z + (x \vdash y) \dashv z - (x\perp y) \dashv z + (x \dashv y) \vdash z + (x \vdash y) \vdash z \\
&\quad -(x\perp y) \vdash z - (x \dashv y)\perp z - (x \vdash y)\perp z + (x\perp y)\perp z \\
&= x \dashv (y \vdash z) + x \vdash (y \dashv z) - x\perp(y \dashv z) + x \vdash (y \vdash z) \\
&\quad +x \vdash (y \vdash z) - x \vdash (y \vdash z) - x\perp(y \vdash z) - x \vdash (y\perp z) + x\perp(y\perp z) \\
&= x \dashv (y*z - y \dashv z + y\perp z) + x \vdash (y \dashv z) - x\perp(y \dashv z) + x \vdash (y \vdash z) \\
&\quad -x\perp(y \vdash z) - x \vdash (y\perp z) + x\perp(y\perp z) \\
&= x \dashv (y*z) - x \dashv (y \dashv z) + x \dashv (y\perp z) + x \vdash (y \dashv z) - x\perp(y \dashv z) \\
&\quad +x \vdash (y \vdash z) - x\perp(y \vdash z) - x \vdash (y\perp z) + x\perp(y\perp z) \\
&= x \vdash (y \dashv z + y \vdash z - y\perp z) - x\perp(y \dashv z + y \vdash z - y\perp z) + x \dashv (y*z) \\
&= x \dashv (y*z) + x \vdash (y*z) - x\perp(y*z) \\
&= x*(y*z).
\end{aligned}
$$

□

Definition 2.6. *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra and let $\lambda \in \mathbb{K}$. If a $\mathbb{K}$-linear map $\Re : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies the Rota-Baxter relation:*

$$\Re(x)\Re(y) = \Re(\Re(x)y + x\Re(y) + \lambda xy)$$

$\forall x, y \in \mathcal{A}$, then $\Re$ is called a Rota-Baxter operator of weight λ and $(\mathcal{A}, \Re)$ is called a Rota-Baxter algebra of weight λ.

Remark 2.7. *If $\Re$ is a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on trialgebras $(\mathcal{T}, \dashv, \perp, \vdash)$. It is also a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on the associative algebra $(\mathcal{T}, *)$.*

Proposition 2.8. *Let $(\mathcal{T}, \dashv, \perp, \vdash)$ be a Rota-Baxter trialgebras of weight 0. Then $(\mathcal{T}, \star)$ is a left-symmetric algebra.*

Proof. For $x, y \in \mathcal{T}$ we have

$$
\begin{aligned}
(x \star y) \star z &= (\Re(x)*y - y*\Re(x)) \star z \\
&= \Re(\Re(x)*y - y*\Re(x))*z - z*\Re(\Re(x)*y - y*\Re(x)) \\
&= \Re(\Re(x)*y)*z - \Re(y*\Re(x))*z - z*\Re(\Re(x)*y) + z*\Re(y*\Re(x))
\end{aligned}
$$

and

$$
\begin{aligned}
x \star (y \star z) &= x \star (\Re(y)*z - z*\Re(y)) \\
&= \Re(x)*(\Re(y)*z - z*\Re(y)) - (\Re(y)*z - z*\Re(y))*\Re(x) \\
&= \Re(x)*(\Re(y)*z) - \Re(x)*(z*\Re(y)) - (\Re(y)*z)*\Re(x) + (z*\Re(y))*\Re(x)
\end{aligned}.
$$

Then

$$
\begin{aligned}
(x \star y) \star z &\quad -x \star (y \star z) - (y \star x) \star z + y \star (x \star z) \\
&= \Re(\Re(x)*y)*z - \Re(y*\Re(x))*z - z*\Re(\Re(x)*y) + z*\Re(y*\Re(x)) \\
&\quad -\Re(x)*(\Re(y)*z) + \Re(x)*(z*\Re(y)) + (\Re(y)*z)*\Re(x) \\
&\quad -(z*\Re(y))*\Re(x) - \Re(\Re(y)*x)*z + \Re(x*\Re(y))*z \\
&\quad +z*\Re(\Re(y)*x) - z*\Re(x*\Re(y)) + \Re(y)*(\Re(x)*z) \\
&\quad -\Re(y)*(z*\Re(x)) - (\Re(x)*z)*\Re(y) + (z*\Re(x))*\Re(y).
\end{aligned}
$$

Using $x * y = x \dashv y + x \vdash y - x \perp y$ and the Rota-Baxter identities.Then associativity leads to
$(x \star y) \star z - x \star (y \star z) - (y \star x) \star z + y \star (x \star z) = 0$. Therefore we obtain $(x, y, z) = (y, x, z)$. $\square$

Proposition 2.9. *Let $(\mathcal{T}, \dashv, \perp, \vdash, \Re)$ be a Rota-Baxter trialgebras of weight -1. Then $(A, \star)$ is an associative algebra.*

Proof. For $x, y \in \mathcal{T}$ we have

$$\begin{aligned} x \star (y \star z) &= \Re(x) * (\Re(y) * z - z * \Re(y) - y * z) \\ &-(\Re(y) * z - z * \Re(y) - y * z) * \Re(x) - x * (\Re(y) * z - z * \Re(y) - y * z) \end{aligned}$$

and

$$\begin{aligned} (x \star y) \star z &= \Re(\Re(x) * y - y * \Re(x) - x * y) * z \\ &-z * \Re(\Re(x) * y - y * \Re(x) - x * y) - (\Re(x) * y - y * \Re(x) - x * y) * z \end{aligned}$$

Then we obtain

$$\begin{aligned} x \star (y \star z) &-(x \star y) \star z \\ &= \Re(x) * (\Re(y) * z - z * \Re(y) - y * z) - (\Re(y) * z - z * \Re(y) - y * z) * \Re(x) \\ &-x * (\Re(y) * z - z * \Re(y) - y * z) - \Re(\Re(x) * y + y * \Re(x) + x * y) * z \\ &+z * \Re(\Re(x) * y + y * \Re(x) + x * y) + (\Re(x) * y + y * \Re(x) + x * y) * z \end{aligned}$$

Then it vanishes using $x * y = x \dashv y + x \vdash y - x \perp y$ and the Rota-Baxter identities. $\square$

3 Classification of low-dimensional associative trialgebras

The classification problem of algebra is one of the important problems of modern algebras. This section describes the classification of associative trialgebras of dimension ≤ 4 over the field $\mathbb{K}$ of characteristic 0. Let $\{e_1, e_2, e_3, \dots, e_n\}$ be the basis of an n-dimensional associative trialgebras $\mathcal{T}$. The product of two elements of the basis $\{e_1, e_2, e_3, \dots, e_n\}$ can be expressed as follows:

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k \quad ; \quad e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^k e_k \quad ; \quad e_i \perp e_j = \sum_{k=1}^{n} \xi_{ij}^k e_k.$$

Where, the tensors (γ_{ij}^k), (δ_{ij}^k) and (ξ_{ij}^k) stand for the families of structure constants of $\mathcal{T}$. Now evaluating the structure constant equations for the tridendriform algebras, we obtain

$$\begin{cases} \sum_{p=1}^{n} (\gamma_{ij}^p \gamma_{pk}^q - \gamma_{jk}^p \gamma_{ip}^q) = 0, & i, j, q \in \{1, \dots, n\} \\ \sum_{p=1}^{n} (\gamma_{ij}^p \gamma_{pk}^q - \delta_{jk}^p \gamma_{ip}^q) = 0, & i, j, q \in \{1, \dots, n\} \\ \sum_{p=1}^{n} (\delta_{ij}^p \gamma_{pk}^q - \gamma_{jk}^p \delta_{ip}^q) = 0, & i, j, q \in \{1, \dots, n\} \\ \sum_{p=1}^{n} (\gamma_{ij}^p \delta_{pk}^q - \delta_{jk}^p \delta_{ip}^q) = 0, & i, j, q \in \{1, \dots, n\} \\ \sum_{p=1}^{n} (\delta_{ij}^p \delta_{pk}^q - \delta_{jk}^p \delta_{ip}^q) = 0, & i, j, q \in \{1, \dots, n\} \\ \sum_{p=1}^{n} (\gamma_{ij}^p \gamma_{pk}^q - \xi_{jk}^p \gamma_{ip}^q) = 0, & i, j, q \in \{1, \dots, n\} \end{cases}$$

$$\begin{cases} \sum_{p=1}^{n}(\xi_{ij}^{p}\gamma_{pk}^{q}-\gamma_{jk}^{p}\xi_{ip}^{q})=0, & i,j,q\in\{1,\dots,n\}\\ \sum_{p=1}^{n}(\gamma_{ij}^{p}\xi_{pk}^{q}-\delta_{jk}^{p}\xi_{ip}^{q})=0, & i,j,q\in\{1,\dots,n\}\\ \sum_{p=1}^{n}(\delta_{ij}^{p}\xi_{pk}^{q}-\xi_{jk}^{p}\delta_{ip}^{q})=0, & i,j,q\in\{1,\dots,n\}\\ \sum_{p=1}^{n}(\xi_{ij}^{p}\delta_{pk}^{q}-\delta_{jk}^{p}\delta_{ip}^{q})=0, & i,j,q\in\{1,\dots,n\}\\ \sum_{p=1}^{n}(\xi_{ij}^{p}\xi_{pk}^{q}-\xi_{jk}^{p}\xi_{ip}^{q})=0, & i,j,q\in\{1,\dots,n\}. \end{cases}$$

Thus Trias can be considered as a subvariety of $3n^3$-dimensional affine space. On Trias the linear matrix group $GL_n(\mathbb{K})$ acts by changing of basis.

Lemma 3.1. *The axioms in Definition 2.1 are respectively equivalent to*

$$\begin{cases} \gamma_{ij}^{p}\gamma_{pk}^{q}-\gamma_{jk}^{p}\gamma_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \gamma_{ij}^{p}\gamma_{pk}^{q}-\delta_{jk}^{p}\gamma_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \delta_{ij}^{p}\gamma_{pk}^{q}-\gamma_{jk}^{p}\delta_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \gamma_{ij}^{p}\delta_{pk}^{q}-\delta_{jk}^{p}\delta_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \delta_{ij}^{p}\delta_{pk}^{q}-\delta_{jk}^{p}\delta_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \gamma_{ij}^{p}\gamma_{pk}^{q}-\xi_{jk}^{p}\gamma_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\} \end{cases} \qquad \begin{cases} \xi_{ij}^{p}\gamma_{pk}^{q}-\gamma_{jk}^{p}\xi_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \gamma_{ij}^{p}\xi_{pk}^{q}-\delta_{jk}^{p}\xi_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \delta_{ij}^{p}\xi_{pk}^{q}-\xi_{jk}^{p}\delta_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \xi_{ij}^{p}\delta_{pk}^{q}-\delta_{jk}^{p}\delta_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}\\ \xi_{ij}^{p}\xi_{pk}^{q}-\xi_{jk}^{p}\xi_{ip}^{q}=0, & i,j,q\in\{1,\dots,n\}. \end{cases}$$

Note that $\mathcal{T}_n^m$ denote m^{th} isomorphism class of associative trialgebra in dimension n.

Theorem 3.2. *Any 2-dimensional real associative trialgebra either is associative or isomorphic to one of the following pairwise non-isomorphic triassociative algebras:*

$$\mathcal{T}_2^1: \begin{array}{lll} e_1\dashv e_2=ae_1, & e_2\vdash e_1=ae_1, & e_1\perp e_1=be_1,\\ e_2\dashv e_2=ae_2, & e_2\vdash e_2=ae_2, & e_1\perp e_2=be_1+ae_2, \end{array}$$

$$\mathcal{T}_2^2: \begin{array}{lll} e_1\dashv e_1=e_1, & e_1\vdash e_1=e_1, & e_1\perp e_2=e_1+ae_2,\\ e_2\dashv e_1=e_2, & e_1\vdash e_2=e_2, & e_2\perp e_2=e_2. \end{array}$$

$$\mathcal{T}_2^3: \; e_2\dashv e_2=e_2, \quad \begin{array}{ll} e_2\vdash e_1=e_1, & e_2\perp e_1=e_1,\\ e_2\vdash e_2=e_2, & e_2\perp e_2=e_2. \end{array}$$

$$\mathcal{T}_2^4: \begin{array}{l} e_1\dashv e_2=e_1,\\ e_2\dashv e_2=e_2, \end{array} \quad e_2\vdash e_2=e_1+e_2, \quad e_2\perp e_2=e_1+e_2.$$

$$\mathcal{T}_2^5: \; e_1\dashv e_1=e_1, \quad \begin{array}{ll} e_2\vdash e_1=e_1, & e_1\perp e_1=e_1,\\ e_2\vdash e_2=e_2, & e_1\perp e_2=e_2. \end{array}$$

$$\mathcal{T}_2^6: \begin{array}{l} e_1\dashv e_1=e_1,\\ e_2\dashv e_1=e_2, \end{array} \quad e_1\vdash e_1=e_1, \quad e_1\perp e_1=e_1.$$

$$\mathcal{T}_2^7: \begin{array}{lll} e_1\dashv e_1=e_1, & e_1\vdash e_1=e_1, & e_1\perp e_1=e_1,\\ e_2\dashv e_1=e_2, & e_1\vdash e_2=e_2, & e_1\perp e_2=e_2. \end{array}$$

$$\mathcal{T}_2^8: \begin{array}{ll} e_1\dashv e_1=ae_1, & e_1\vdash e_1=ae_1,\\ e_2\dashv e_1=ae_2, & e_1\vdash e_2=ae_2, \end{array} \quad e_1\perp e_1=ae_1+be_2.$$

Proof. Let $\mathcal{T}$ be a two-dimensional vector space. To determine an associative trialgebras structure on $\mathcal{T}$, we consider $\mathcal{T}$ with respect to one associative trialgebra operation. Let

$\mathcal{A} = (\mathcal{T}, \dashv)$ be the algebra

$$e_1 \dashv e_1 = e_1,\ e_2 \dashv e_1 = e_2$$

The multiplication operations $\vdash, \perp$ *in* $\mathcal{T}$, we define as follows:

$$\begin{array}{lll} e_1 \vdash e_1 = \alpha_1 e_1 + \alpha_2 e_2, & e_2 \vdash e_2 = \alpha_7 e_1 + \alpha_8 e_2, & e_2 \perp e_1 = \beta_5 e_1 + \beta_6 e_2, \\ e_1 \vdash e_2 = \alpha_3 e_1 + \alpha_4 e_2, & e_1 \perp e_1 = \beta_1 e_1 + \beta_2 e_2, & e_2 \perp e_2 = \beta_7 e_1 + \beta_8 e_2. \\ e_2 \vdash e_1 = \alpha_5 e_1 + \alpha_6 e_2, & e_1 \perp e_2 = \beta_3 e_1 + \beta_4 e_2, & \end{array}$$

Now verifying associative trialgebra axioms, we get several constraints for the coefficients α_i, β_i where $1 \le i \le 8$.
Applying $(e_1 \vdash e_1) \dashv e_1 = e_1 \vdash (e_1 \dashv e_1)$, we get $(\alpha_1 e_1 + \alpha_2 e_2) \dashv e_1 = e_1 \vdash e_1$ and then $\alpha_1 e_1 = e_1$. Therefore $\alpha_1 = 1$. The verification, $(e_1 \vdash e_1) \dashv e_1 = e_1 \vdash (e_1 \vdash e_1)$ leads to $(e_1 + \alpha_2 e_2) \vdash e_1 = e_1 \vdash e_1$. We have $e_1 + \alpha_2 e_2 = e_1$. Hence we have $\alpha_2 = 0$. Consider $(e_1 \perp e_1) \dashv e_1 = e_1 \perp (e_1 \dashv e_1)$. It implies that $(\beta_1 e_1 + \beta_2 e_2) \dashv e_1 = e_1 \perp e_1$, therefore $\beta_1 = 1$ and $\beta_2 = 0$. then $\mathcal{A} = (\mathcal{T}, \dashv)$ it is isomorphic to $\mathcal{T}_2^6$. The other associative trialgebras of the list of Theorem 3.2 can be obtained by minor modifications of the observation above. □

Theorem 3.3. *Any 3-dimensional real associative trialgebra either is associative or isomorphic to one of the following pairwise non-isomorphic associative trialgebras :*

$$\mathcal{T}_3^1: \quad \begin{array}{l} e_1 \dashv e_2 = e_3, \\ e_2 \dashv e_1 = e_3, \\ e_2 \dashv e_3 = e_3, \end{array} \quad \begin{array}{l} e_1 \vdash e_2 = e_3, \\ e_2 \vdash e_2 = e_3, \end{array} \quad \begin{array}{l} e_1 \perp e_1 = e_3, \\ e_1 \perp e_2 = e_3, \\ e_2 \perp e_2 = e_3. \end{array}$$

$$\mathcal{T}_3^2: \quad \begin{array}{l} e_1 \dashv e_2 = e_3, \\ e_2 \dashv e_1 = e_3, \\ e_2 \dashv e_3 = e_3, \end{array} \quad \begin{array}{l} e_1 \vdash e_2 = e_3, \\ e_2 \vdash e_1 = e_3, \\ e_2 \vdash e_2 = e_3 \end{array} \quad \begin{array}{l} e_1 \perp e_1 = e_3, \\ e_1 \perp e_2 = e_3, \\ e_2 \perp e_2 = e_3. \end{array}$$

$$\mathcal{T}_3^3: \quad e_2 \dashv e_2 = e_1, \quad e_2 \vdash e_2 = e_1, \quad \begin{array}{l} e_2 \perp e_2 = e_3, \\ e_2 \perp e_3 = e_1 + e_3. \end{array}$$

$$\mathcal{T}_3^4: \quad e_3 \dashv e_3 = e_1, \quad e_3 \vdash e_3 = e_1, \quad \begin{array}{l} e_3 \perp e_2 = e_1 + e_2, \\ e_3 \perp e_3 = e_1 + e_2. \end{array}$$

$$\mathcal{T}_3^5: \quad e_2 \dashv e_2 = e_1 + e_3, \quad e_2 \vdash e_2 = e_1 + e_3, \quad e_2 \perp e_2 = e_1 + e_2.$$

$$\mathcal{T}_3^6: \quad \begin{array}{l} e_1 \dashv e_3 = e_2, \\ e_3 \dashv e_1 = e_2, \\ e_3 \dashv e_3 = e_2, \end{array} \quad \begin{array}{l} e_1 \vdash e_1 = e_2, \\ e_1 \vdash e_3 = e_2, \\ e_3 \vdash e_1 = e_2, \\ e_3 \vdash e_3 = e_2, \end{array} \quad \begin{array}{l} e_3 \perp e_1 = e_2, \\ e_3 \perp e_3 = e_2. \end{array}$$

$$\mathcal{T}_3^7: \quad e_1 \dashv e_1 = e_2 + e_3, \quad e_1 \vdash e_1 = e_2 + e_3, \quad e_1 \perp e_1 = e_2 + e_3.$$

$$\mathcal{T}_3^8: \quad \begin{array}{l} e_2 \dashv e_2 = e_1, \\ e_2 \dashv e_3 = e_1, \\ e_3 \dashv e_2 = e_1, \\ e_3 \dashv e_3 = e_1, \end{array} \quad \begin{array}{l} e_2 \vdash e_2 = e_1, \\ e_2 \vdash e_3 = e_1, \\ e_3 \vdash e_2 = e_1, \end{array} \quad \begin{array}{l} e_2 \perp e_2 = e_1, \\ e_2 \perp e_3 = e_1, \\ e_3 \perp e_2 = e_1. \end{array}$$

$\mathcal{T}_3^9$: $e_2 \dashv e_2 = e_3$, $e_2 \vdash e_2 = e_3$, $e_2 \perp e_1 = e_1 + e_3$, $e_2 \perp e_2 = e_1 + e_3$.

$\mathcal{T}_3^{10}$: $e_2 \dashv e_2 = e_1 + e_3$, $e_2 \vdash e_2 = e_1 + e_3$, $e_2 \perp e_2 = e_1 + e_3$.

$\mathcal{T}_3^{11}$: $e_2 \dashv e_1 = e_3$, $e_2 \dashv e_2 = e_3$, $e_2 \vdash e_1 = e_3$, $e_2 \vdash e_2 = e_3$, $e_2 \perp e_1 = e_3$, $e_2 \perp e_2 = e_3$.

$\mathcal{T}_3^{12}$: $e_1 \dashv e_1 = e_3$, $e_1 \dashv e_2 = e_3$, $e_2 \dashv e_1 = e_3$, $e_1 \vdash e_2 = e_3$, $e_2 \vdash e_1 = e_3$, $e_2 \vdash e_2 = e_3$. $e_2 \perp e_1 = e_3$, $e_2 \perp e_2 = e_3$.

Proof. Let $\mathcal{T}$ be a three-dimensional vector space. To determine an associative trialgebras structure on $\mathcal{T}$, we consider $\mathcal{T}$ with respect to one associative trialgebra operation. Let $\mathcal{B} = (\mathcal{T}, \dashv)$ be the algebra

$$e_2 \dashv e_1 = e_3,\ e_2 \dashv e_2 = e_3$$

The multiplication operations $\vdash, \perp$ *in* $\mathcal{T}$. We use the same method of the proof of the Theorem 3.2.

Then $\mathcal{B} = (\mathcal{T}, \dashv)$ it is isomorphic to $\mathcal{T}_3^{11}$. The other associative trialgebras of the list of Theorem 3.3 can be obtained by minor modification of the observation above. □

Theorem 3.4. *Any 4-dimensional real associative trialgebra either is associative or isomorphic to one of the following pairwise non-isomorphic associative tialgebras:*

$\mathcal{T}_4^1$: $e_1 \dashv e_1 = e_2 + e_4$, $e_1 \dashv e_3 = e_2 + e_4$, $e_3 \dashv e_1 = e_4$, $e_1 \vdash e_1 = e_2 + e_4$, $e_1 \vdash e_3 = e_2 + e_4$, $e_3 \vdash e_1 = e_4$, $e_1 \perp e_1 = e_2 + e_4$, $e_1 \perp e_3 = e_4$, $e_3 \perp e_3 = e_2$,

$\mathcal{T}_4^2$: $e_1 \dashv e_1 = e_2 + e_4$, $e_1 \dashv e_3 = e_2 + e_4$, $e_3 \dashv e_1 = e_2 + e_4$, $e_1 \vdash e_1 = e_2 + e_4$, $e_1 \vdash e_3 = e_2 + e_4$, $e_3 \vdash e_1 = e_2 + e_4$, $e_1 \perp e_1 = e_2 + e_4$, $e_1 \perp e_3 = e_2 + e_4$, $e_3 \perp e_1 = e_2 + e_4$, $e_3 \perp e_3 = e_2$.

$\mathcal{T}_4^3$: $e_1 \dashv e_1 = e_2 + e_4$, $e_1 \dashv e_3 = e_2 + e_4$, $e_3 \dashv e_1 = e_2 + e_4$, $e_1 \vdash e_1 = e_2 + e_4$, $e_1 \vdash e_3 = e_2 + e_4$, $e_3 \vdash e_1 = e_2 + e_4$, $e_1 \perp e_1 = e_2 + e_4$, $e_1 \perp e_3 = e_2 + e_4$, $e_3 \perp e_3 = e_4$.

$\mathcal{T}_4^4$: $e_1 \dashv e_2 = e_4$, $e_2 \dashv e_1 = e_4$, $e_2 \dashv e_2 = e_4$, $e_2 \vdash e_1 = e_4$, $e_2 \vdash e_2 = e_4$, $e_3 \vdash e_1 = e_4$, $e_1 \perp e_2 = e_4$, $e_2 \perp e_1 = e_4$, $e_1 \perp e_2 = e_4$, $e_2 \perp e_1 = e_4$,

$\mathcal{T}_4^5$: $e_1 \dashv e_2 = e_4$, $e_2 \dashv e_1 = e_4$, $e_2 \dashv e_2 = e_4$, $e_2 \vdash e_1 = e_4$, $e_2 \vdash e_2 = e_4$, $e_3 \vdash e_1 = e_4$, $e_1 \perp e_1 = e_4$, $e_2 \perp e_1 = e_4$, $e_3 \perp e_3 = e_4$.

$$\mathcal{T}_4^6: \begin{array}{lll} e_3 \dashv e_4 = e_1 + e_2, & e_3 \vdash e_4 = e_1 + e_2, & e_3 \perp e_4 = e_1 + e_2, \\ e_4 \dashv e_3 = e_1 + e_2, & e_4 \vdash e_3 = e_1 + e_2, & e_4 \perp e_3 = e_1 + e_2, \\ e_4 \dashv e_4 = e_1 + e_2, & e_4 \vdash e_4 = e_1 + e_2, & e_4 \perp e_4 = e_1 + e_2. \end{array}$$

$$\mathcal{T}_4^7: \begin{array}{lll} e_2 \dashv e_2 = e_1 + e_3, & e_2 \vdash e_2 = e_1 + e_3, & e_2 \perp e_2 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, & e_2 \vdash e_4 = e_1 + e_3, & e_2 \perp e_4 = e_1 + e_3, \\ e_4 \dashv e_2 = e_1 + e_3, & e_4 \vdash e_2 = e_1 + e_3, & e_4 \perp e_2 = e_1 + e_3. \end{array}$$

$$\mathcal{T}_4^8: \begin{array}{l} e_2 \dashv e_2 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, \\ e_4 \dashv e_2 = e_1 + e_3, \\ e_4 \dashv e_4 = e_1 + e_3, \end{array} \quad \begin{array}{l} e_2 \vdash e_2 = e_1 + e_3, \\ e_2 \vdash e_4 = e_1 + e_3, \\ e_4 \vdash e_2 = e_1 + e_3, \end{array} \quad \begin{array}{l} e_2 \perp e_2 = e_1 + e_3, \\ e_2 \perp e_4 = e_1 + e_3, \\ e_4 \perp e_4 = e_1 + e_3. \end{array}$$

$$\mathcal{T}_4^9: \begin{array}{l} e_2 \dashv e_2 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, \\ e_4 \dashv e_2 = e_1 + e_3, \end{array} \quad \begin{array}{l} e_4 \dashv e_4 = e_1 + e_3, \\ e_2 \vdash e_2 = e_1 + e_3, \\ e_2 \vdash e_4 = e_1 + e_3, \\ e_4 \vdash e_2 = e_1 + e_3, \\ e_4 \vdash e_4 = e_3, \end{array} \quad \begin{array}{l} e_2 \perp e_2 = e_1 + e_3, \\ e_2 \perp e_4 = e_1 + e_3, \\ e_4 \perp e_2 = e_1 + e_3, \\ e_4 \perp e_4 = e_1 \end{array}$$

$$\mathcal{T}_4^{10}: \begin{array}{lll} e_2 \dashv e_2 = e_1 + e_3, & e_2 \vdash e_2 = e_1 + e_3, & e_2 \perp e_4 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, & e_4 \vdash e_2 = e_1 + e_3, & e_4 \perp e_2 = e_1 + e_3, \\ e_4 \dashv e_4 = e_1 + e_3, & e_4 \vdash e_4 = e_3, & e_4 \perp e_4 = e_1. \end{array}$$

$$\mathcal{T}_4^{11}: \begin{array}{lll} e_2 \dashv e_2 = e_1 + e_3, & e_2 \vdash e_2 = e_1 + e_3, & e_2 \perp e_2 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, & e_2 \vdash e_4 = e_1 + e_3, & e_2 \perp e_4 = e_1 + e_3, \\ e_4 \dashv e_2 = e_1 + e_3, & e_4 \vdash e_2 = e_1 + e_3, & e_4 \perp e_2 = e_1 + e_3. \end{array}$$

$$\mathcal{T}_4^{12}: \begin{array}{l} e_2 \dashv e_2 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, \\ e_4 \dashv e_2 = e_1 + e_3, \\ e_4 \dashv e_4 = e_1 + e_3, \end{array} \quad \begin{array}{l} e_2 \vdash e_2 = e_1 + e_3, \\ e_2 \vdash e_4 = e_1 + e_3, \\ e_4 \vdash e_2 = e_1 + e_3, \end{array} \quad \begin{array}{l} e_2 \perp e_2 = e_1 + e_3, \\ e_2 \perp e_4 = e_1 + e_3, \\ e_4 \perp e_2 = e_1 + e_3. \end{array}$$

$$\mathcal{T}_4^{13}: \begin{array}{lll} e_2 \dashv e_1 = e_4, & e_1 \vdash e_3 = e_4, & e_1 \perp e_1 = e_4, \\ e_2 \dashv e_2 = e_4, & e_2 \vdash e_2 = e_4, & e_1 \perp e_3 = e_4, \\ e_3 \dashv e_3 = e_4, & e_3 \vdash e_1 = e_4, & e_3 \perp e_3 = e_4. \end{array}$$

$$\mathcal{T}_4^{14}: \begin{array}{lll} e_2 \dashv e_2 = e_1 + e_3, & e_2 \vdash e_2 = e_1 + e_3, & e_2 \perp e_2 = e_1 + e_3, \\ e_2 \dashv e_4 = e_1 + e_3, & e_2 \vdash e_4 = e_1 + e_3, & e_2 \perp e_4 = e_1 + e_3, \\ e_4 \dashv e_2 = e_1 + e_3, & e_4 \vdash e_2 = e_1 + e_3, & e_4 \perp e_2 = e_1 + e_3, \\ e_4 \dashv e_4 = e_1 + e_3, & e_4 \vdash e_4 = e_3, & e_4 \perp e_4 = e_1. \end{array}$$

$$\mathcal{T}_4^{15}: \begin{array}{lll} e_2 \dashv e_1 = e_3, & e_1 \vdash e_1 = e_3, & e_2 \perp e_1 = e_3, \\ e_2 \dashv e_2 = e_3, & e_1 \vdash e_4 = e_3, & e_2 \perp e_2 = e_3, \\ e_4 \dashv e_1 = e_3, & e_2 \vdash e_1 = e_3, & e_4 \perp e_1 = e_3, \\ e_4 \dashv e_2 = e_3, & e_2 \vdash e_4 = e_3, & e_4 \perp e_4 = e_3. \end{array}$$

$$\mathcal{T}_4^{16}: \begin{array}{l} e_1 \dashv e_1 = e_2 + e_4, \\ e_3 \dashv e_1 = e_2 + e_4, \\ e_3 \dashv e_3 = e_2 + e_4, \end{array} \quad \begin{array}{l} e_1 \vdash e_1 = e_2 + e_4, \\ e_1 \vdash e_3 = e_2 + e_4, \\ e_3 \vdash e_1 = e_2 + e_4, \\ e_3 \vdash e_3 = e_4, \end{array} \quad \begin{array}{l} e_1 \perp e_1 = e_2, \\ e_1 \perp e_3 = e_4, \\ e_3 \perp e_1 = e_2, \\ e_3 \perp e_3 = e_2 + e_4. \end{array}$$

Proof. Let $\mathcal{T}$ be a three-dimensional vector space. To determine an associative trialgebra structure on $\mathcal{T}$, we consider $\mathcal{T}$ with respect to one associative trialgebra operation. Let $\mathcal{C} = (\mathcal{T}, \dashv)$ be the algebra

$$e_1 \dashv e_2 = e_4,\ e_2 \dashv e_1 = e_4, e_2 \dashv e_2 = e_4.$$

The multiplication operations $\vdash, \perp in\ \mathcal{T}$. We use the same method of the proof of the Theorem 3.2.

Then $\mathcal{C} = (\mathcal{T}, \dashv)$ it is isomorphic to $\mathcal{T}_4^4$. The other associative trialgebras of the list of Theorem 3.4 can be obtained by minor modification of the observation. □

4 Derivations of low-dimensional associative trialgebras

Definition 4.1. *A derivation of the associative trialgebras $\mathcal{T}$ is a linear transformation : $\kappa : \mathcal{T} \to \mathcal{T}$ satisfying*

$$\begin{cases} \kappa(x \dashv y) = \kappa(x) \dashv y + x \dashv \kappa(y) \\ \kappa(x \vdash y) = \kappa(x) \vdash y + x \vdash \kappa(y) \\ \kappa(x \perp y) = \kappa(x) \perp y + x \perp \kappa(y) \end{cases} \tag{4}$$

for all $x, y \in \mathcal{T}$.

Let $(\mathcal{T}, \dashv, \vdash, \perp)$ be an $n-$dimensional triassocitive algebra with basis $\{e_i\}$ $(1 \leq i \leq n)$ and let κ be a derivation on $\mathcal{T}$ For any $i, j, k \in \mathbb{N}$, $1 \leq i, j, k \leq n$, let us put

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k\,;\ e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^k e_k\,;\ e_i \perp e_j = \sum_{k=1}^{n} \xi_{ij}^k e_k, \text{ and } \kappa(e_i) = \sum_{j=1}^{n} \kappa_{ji} e_j.$$

Then, in term of basis elements, equation (4) is equivalent to

$$\begin{cases} \sum_{q=1}^{n} (\gamma_{ij}^k d_{qk} - \kappa_{ki}\gamma_{kj}^q - \kappa_{kj}\gamma_{ik}^q) = 0,\ i, j, q \in \{1, \ldots, n\} \\ \sum_{q=1}^{n} (\delta_{ij}^k d_{qk} - \kappa_{ki}\delta_{kj}^q - \kappa_{kj}\delta_{ik}^q) = 0,\ i, j, q \in \{1, \ldots, n\} \\ \sum_{q=1}^{n} (\xi_{ij}^k \kappa_{qk} - \kappa_{ki}\xi_{kj}^q - \kappa_{kj}\xi_{ik}^q) = 0,\ i, j, q \in \{1, \ldots, n\}\,. \end{cases}$$

Theorem 4.2. *The derivations of two-dimensional associative trialgebras have the following form :*

$$\mathcal{T}_2^3 : \begin{pmatrix} \kappa_{11} & 0 \\ 0 & 0 \end{pmatrix};\ \mathcal{T}_2^4 : \begin{pmatrix} \kappa_{11} & 0 \\ -\kappa_{11} & 0 \end{pmatrix};\ \mathcal{T}_2^6 : \begin{pmatrix} 0 & 0 \\ 0 & \kappa_{22} \end{pmatrix};\mathcal{T}_2^7 : \begin{pmatrix} \kappa_{11} & 0 \\ 0 & \kappa_{11} \end{pmatrix};\ \mathcal{T}_2^8 : \begin{pmatrix} \frac{1}{2}\kappa_{22} & 0 \\ \kappa_{21} & \kappa_{22} \end{pmatrix}.$$

Theorem 4.3. *The derivations of three-dimensional associative trialgebras have the following form :*

$$\mathcal{T}_3^3 : \begin{pmatrix} 0 & 0 & 0 \\ \kappa_{21} & 0 & \kappa_{13} \\ 0 & 0 & \kappa_{13} \end{pmatrix};\ \mathcal{T}_3^4 : \begin{pmatrix} 0 & \kappa_{23} & \kappa_{13} \\ 0 & \kappa_{23} & \kappa_{23} \\ 0 & 0 & 0 \end{pmatrix};\ \mathcal{T}_3^5 : \begin{pmatrix} -\kappa_{13} & \kappa_{13} & \kappa_{13} \\ 0 & 0 & 0 \\ -\kappa_{33} & \kappa_{33} & \kappa_{33} \end{pmatrix};$$

$$\mathcal{T}_3^6 : \begin{pmatrix} \kappa_{33} & 0 & 0 \\ \kappa_{21} & 2\kappa_{33} & \kappa_{33} \\ 0 & 0 & \kappa_{33} \end{pmatrix}; \mathcal{T}_3^7 : \begin{pmatrix} \kappa_{11} & 0 & 0 \\ \kappa_{21} & -\kappa_{23}+2\kappa_{11} & \kappa_{23} \\ \kappa_{31} & -\kappa_{33}+2\kappa_{11} & \kappa_{33} \end{pmatrix}; \mathcal{T}_3^8 : \begin{pmatrix} 2\kappa_{33} & \kappa_{12} & \kappa_{13} \\ 0 & \kappa_{33} & 0 \\ 0 & 0 & \kappa_{33} \end{pmatrix};$$

$$\mathcal{T}_3^9 : \begin{pmatrix} \kappa_{13} & \kappa_{31} & 0 \\ 0 & 0 & 0 \\ \kappa_{31} & \kappa_{32} & 0 \end{pmatrix}; \mathcal{T}_3^{10} : \begin{pmatrix} -\kappa_{13}+\kappa_{22} & \kappa_{12} & \kappa_{13} \\ 0 & \kappa_{22} & \\ -\kappa_{33}+\kappa_{22} & \kappa_{32} & \kappa_{33} \end{pmatrix};$$

$$\mathcal{T}_3^{11} : \begin{pmatrix} \kappa_{33}-2\kappa_{22} & \kappa_{33}-2\kappa_{22} & 0 \\ 0 & \kappa_{22} & 0 \\ \kappa_{31} & \kappa_{32} & \kappa_{33} \end{pmatrix}; \mathcal{T}_3^{12} : \begin{pmatrix} \kappa_{11} & 0 & 0 \\ 0 & \kappa_{11} & 0 \\ \kappa_{31} & \kappa_{32} & 2\kappa_{11} \end{pmatrix}$$

Proof. From Theorem 4.3, we provide the proof only for one case to illustrate the approach used, the other cases can be carried out similarly with or no modification(s). Let's consider $\mathcal{T}_3^8$. Applying the systems of equations (4). we get $\kappa_{21} = \kappa_{23} = \kappa_{31} = \kappa_{32} = 0, \quad \kappa_{11} = 2\kappa_{33}, \quad \kappa_{22} = \kappa_{33}$. Hence, the derivations of $\mathcal{T}_3^8$ are given as follows
$\kappa_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \kappa_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \kappa_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is basis of $Der(\mathcal{T}_3^8)$ and $\mathrm{Dim}Der(\mathcal{T}_3^8) = 3$. The centroids of the remaining parts of three-dimension associative trialgebras can be carried out in similar manner as shown above. □

Theorem 4.4. *The derivations of four-dimensional associative trialgebras have the following form :*

$$\mathcal{T}_4^1 : \begin{pmatrix} \kappa_{33} & 0 & 0 & 0 \\ \kappa_{21} & 2\kappa_{33} & \kappa_{23} & 0 \\ 0 & 0 & \kappa_{33} & 0 \\ \kappa_{41} & 0 & \kappa_{43} & 2\kappa_{33} \end{pmatrix}; \mathcal{T}_4^2 : \begin{pmatrix} \kappa_{11} & 0 & 0 & 0 \\ \kappa_{21} & \kappa_{11} & \kappa_{23} & 0 \\ 0 & 0 & \kappa_{11} & 0 \\ \kappa_{41} & 0 & \kappa_{43} & \kappa_{11} \end{pmatrix}; \mathcal{T}_4^3 : \begin{pmatrix} \kappa_{11} & \kappa_{12} & 0 & \kappa_{14} \\ 0 & \kappa_{11} & 0 & 0 \\ 0 & \kappa_{32} & \kappa_{11} & \kappa_{34} \\ 0 & 0 & 0 & 2\kappa_{11} \end{pmatrix}$$

$$\mathcal{T}_4^4 : \begin{pmatrix} \kappa_{33} & 0 & 0 & 0 \\ 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & 2\kappa_{33} \end{pmatrix}; \mathcal{T}_4^5 : \begin{pmatrix} \kappa_{33} & 0 & 0 & 0 \\ 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & 2\kappa_{33} \end{pmatrix}; \mathcal{T}_4^6 : \begin{pmatrix} \kappa_{33} & 0 & 0 & 0 \\ 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & 2\kappa_{33} \end{pmatrix}$$

$$\mathcal{T}_4^7 : \begin{pmatrix} \kappa_{33} & 0 & 0 & 0 \\ 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & 2\kappa_{33} \end{pmatrix}; \mathcal{T}_4^8 : \begin{pmatrix} -\kappa_{13}+2\kappa_{44} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ 0 & \kappa_{44} & 0 & 0 \\ -\kappa_{33}+2\kappa_{44} & \kappa_{32} & \kappa_{33} & \kappa_{34} \\ 0 & 0 & 0 & \kappa_{44} \end{pmatrix};$$

$$\mathcal{T}_4^9 : \begin{pmatrix} 2\kappa_{44} & \kappa_{12} & 0 & \kappa_{14} \\ 0 & \kappa_{44} & 0 & 0 \\ 0 & 0 & 2\kappa_{22} & \kappa_{34} \\ 0 & 0 & 0 & \kappa_{44} \end{pmatrix}; \mathcal{T}_4^{10} : \begin{pmatrix} 2\kappa_{44} & \kappa_{12} & 0 & \kappa_{14} \\ 0 & \kappa_{44} & 0 & 0 \\ 0 & 0 & 2\kappa_{22} & \kappa_{34} \\ 0 & 0 & 0 & \kappa_{44} \end{pmatrix};$$

$$\mathcal{T}_4^{11} : \begin{pmatrix} \kappa_{11} & 0 & 0 & 0 \\ \kappa_{21} & \frac{1}{2}\kappa_{11} & \kappa_{23} & 0 \\ 0 & 0 & \kappa_{11} & 0 \\ \kappa_{41} & 0 & \kappa_{43} & \frac{1}{2}\kappa_{11} \end{pmatrix}; \mathcal{T}_4^{12} : \begin{pmatrix} \kappa_{11} & 0 & 0 & 0 \\ \kappa_{21} & \frac{1}{2}\kappa_{11} & \kappa_{23} & 0 \\ 0 & 0 & \kappa_{11} & 0 \\ \kappa_{41} & 0 & \kappa_{43} & \frac{1}{2}\kappa_{11} \end{pmatrix};$$

$$\mathcal{T}_4^{13} : \begin{pmatrix} \kappa_{33} & 0 & 0 & 0 \\ 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & 2\kappa_{33} \end{pmatrix}; \mathcal{T}_4^{14} : \begin{pmatrix} 2\kappa_{44} & \kappa_{12} & 0 & \kappa_{14} \\ 0 & \kappa_{44} & 0 & 0 \\ 0 & \kappa_{32} & 2\kappa_{44} & \kappa_{34} \\ 0 & 0 & 0 & \kappa_{44} \end{pmatrix};$$

$$\mathcal{T}_4^{15}: \begin{pmatrix} \kappa_{44} & 0 & 0 & 0 \\ 0 & \kappa_{44} & 0 & 0 \\ \kappa_{31} & \kappa_{32} & 2\kappa_{44} & \kappa_{34} \\ 0 & 0 & 0 & \kappa_{44} \end{pmatrix}; \mathcal{T}_4^{16}: \begin{pmatrix} \kappa_{33} & \kappa_{12} & 0 & 0 \\ 0 & 2\kappa_{33} & 0 & 0 \\ 0 & \kappa_{32} & \kappa_{33} & 0 \\ \kappa_{41} & 0 & \kappa_{43} & 2\kappa_{33} \end{pmatrix}.$$

Proof. From Theorem 4.4, we provide the proof only for one case to illustrate the approach used, the other cases can be carried out similarly with or no modification(s). Let's consider $\mathcal{T}_4^6$. Applying the systems of equations (4). we get $\kappa_{12} = \kappa_{13} = \kappa_{14} = \kappa_{21} = \kappa_{23} = \kappa_{24} = \kappa_{31} = \kappa_{32} = \kappa_{34} = \kappa_{41} = \kappa_{42} = \kappa_{43} = 0$, $\kappa_{22} = \kappa_{11}$, $\kappa_{33} = \kappa_{11}$ $\kappa_{44} = 2\kappa_{11}$. Hence, the derivations of $\mathcal{T}_4^6$ are given as follows

$$\kappa_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \kappa_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \kappa_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \kappa_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is basis of $Der(\mathcal{T})$ and Dim$Der(\mathcal{T}) = 4$. The centroids of the remaining four-dimensional associative trialgebras can be determined in a similar manner as shown above. □

Corollary 4.5.

- *The dimensions of the derivations of two-dimensional associative trialgebras range between zero and two.*
- *The dimensions of the derivations of three-dimensional associative trialgebras range between zero and five.*
- *The dimensions of the derivations of four-dimensional associative trialgebras range between four and five.*

5 Centroids of low-dimensional associative trialgebras

5.1 Properties of centroids of associative trialgebras

In this section, we state the following results on properties of centroids of associative trialgebra $\mathcal{T}$.

Definition 5.1. *Let $\mathcal{H}$ be a nonempty subset of $\mathcal{T}$. The subset*

$$Z_{\mathcal{T}}(\mathcal{H}) = \{x \in \mathcal{H} | x \bullet \mathcal{H} = \mathcal{H} \bullet x = 0\}, \quad (5)$$

is said to be the centralizer of $\mathcal{H}$ in $\mathcal{T}$ where the $\bullet$ is $\dashv, \vdash$ and $\perp$, respectively.

Definition 5.2. *Let $\mathcal{T}$ be an arbitrary associative trialgebra over a field $\mathbb{K}$. The left, right and middle centroids $\Gamma_{\mathbb{K}}^{\dashv}(\mathcal{T})$, $\Gamma_{\mathbb{K}}^{\vdash}(\mathcal{T})$ and $\Gamma_{\mathbb{K}}^{\perp}(\mathcal{T})$ of $\mathcal{T}$ are the spaces of $\mathbb{K}$-linear transformations on $\mathcal{T}$ given by*

$$\Gamma_{\mathbb{K}}^{\bullet}(\mathcal{T}) = \{\psi \in End_{\mathbb{K}}(\mathcal{T}) | \psi(x \bullet y) = x \bullet \psi(y) = \psi(x) \bullet y \quad \text{for all} \quad x, y \in \mathcal{T}\}, \quad (6)$$

where the $\bullet$ is $\dashv, \vdash$ and $\perp$ respectively.

Definition 5.3. *Let $\psi \in End_{\mathbb{K}}(\mathcal{T})$. If $\psi(\mathcal{T}) \subseteq C(\mathcal{T})$ and $\psi(\mathcal{T}^2) = 0$ then ψ is called a central derivation. The set of all central derivations of $\mathcal{T}$ is denoted by $C(\mathcal{T})$.*

Proposition 5.4. *Let $(\mathcal{T}, \vdash, \dashv, \perp)$ be an associative trialgebra. Then*

i) $\Gamma(\mathcal{T})Der(\mathcal{T}) \subseteq Der(\mathcal{T})$.

ii) $[\Gamma(\mathcal{T}), Der(\mathcal{T})] \subseteq \Gamma(\mathcal{T})$.

iii) $[\Gamma(\mathcal{T}), \Gamma(\mathcal{T})]\,(\mathcal{T}) \subseteq \Gamma(\mathcal{T})$ *and* $[\Gamma(\mathcal{T}), \Gamma(\mathcal{T})]\,(\mathcal{T}^2) = 0$.

Proof. The proof of parts $i) - iii)$ is straightforward by using definitions of derivation and centroid. □

Proposition 5.5. *Let $\mathcal{T}$ be an associative trialgebra and $\varphi \in \Gamma(\mathcal{T})$, $\kappa \in Der(\mathcal{T})$. Then $\varphi \circ \kappa$ is a derivation of $\mathcal{T}$.*

Proof. Indeed, if $x, y \in \mathcal{T}$, then

$$\begin{aligned}(\varphi \circ \kappa)(x \bullet y) &= \varphi(\kappa(x) \bullet y + x \bullet \kappa(y)) \\ &= \varphi(\kappa(x) \bullet y) + \varphi(x \bullet \kappa(y)) = (\varphi \circ \kappa)(x) \bullet y + x \bullet (\varphi \circ \kappa)(y)\end{aligned}$$

where the $\bullet$ is $\dashv, \vdash$ and $\perp$ respectively. □

Proposition 5.6. *Let $\mathcal{T}$ be an associative trialgebra over a field $\mathbb{K}$. Then $C(\mathcal{T}) = \Gamma(\mathcal{T}) \cap Der(\mathcal{T})$.*

Proof. If $\psi \in \Gamma(\mathcal{T}) \cap Der(\mathcal{T})$ the by definition of $\Gamma(\mathcal{T})$ and $Der(\mathcal{T})$ we have

$\psi(x \bullet y) = \psi(x) \bullet y + x \bullet \psi(y)$ and $\psi(x \bullet y) = \psi(x) \circ y = x \circ \psi(y)$ for $x, y \in \mathcal{T}$. The yieds $\psi(\mathcal{T}^2) = 0$ and $\psi(\mathcal{T}) \subseteq C(\mathcal{T})$ i.e $\Gamma(\mathcal{T}) \cap Der(\mathcal{T}) \subseteq C(\mathcal{T})$. The inverse is obvious since $C(\mathcal{T})$ is in both $\Gamma(\mathcal{T})$ and $Der(\mathcal{T})$, where the $\bullet$ is $\dashv, \vdash$ and $\perp$ respectively. □

Proposition 5.7. *Let $(\mathcal{T}, \vdash, \dashv, \perp)$ be an associative trialgebra. Then for any $\kappa \in Der(\mathcal{T})$ and $\varphi \in \Gamma(\mathcal{T})$.*

(i) The composition $\kappa \circ \varphi$ is in $\Gamma(\mathcal{T})$ if and only if $\varphi \circ \kappa$ is a central derivation of $\mathcal{T}$.

(ii) The composition $\kappa \circ \varphi$ is a derivation of $\mathcal{T}$ if and only if $[\kappa, \varphi]$ is a central derivation of $\mathcal{T}$.

Proof. i) For any $\varphi \in \Gamma(\mathcal{T})$, $\kappa \in Der(\mathcal{T})$, $\forall\, x, y \in \mathcal{T}$. We have

$$\begin{aligned}\kappa \circ \varphi(x \bullet y) = \kappa \circ \varphi(x) \bullet y &= \kappa \circ \varphi(x) \bullet y + \varphi(x) \bullet \kappa(y) \\ &= \kappa \circ \varphi(x) \bullet y + \varphi \circ \kappa(x \bullet y) - \varphi \circ \kappa(x) \bullet y.\end{aligned}$$

Therefore $(\kappa \circ \varphi - \varphi \circ \kappa)(x \bullet y) = (\kappa \bullet \varphi - \varphi \circ \kappa)(x) \bullet y$.

ii) Let $\kappa \circ \varphi \in Der(\mathcal{T})$, using $[\kappa, \varphi] \in \Gamma(\mathcal{T})$, we get

$$[\kappa, \varphi]\,(x \bullet y) = ([\kappa, \varphi]\,(x)) \bullet y = x \bullet ([\kappa, \varphi]\,(y)) \tag{7}$$

On the other hand $[\kappa, \varphi] = \kappa \circ \varphi - \varphi \circ \kappa$ and $\kappa \circ \varphi, \varphi \circ \kappa \in Der(\mathcal{T})$. Therefore,

$$[\kappa, \varphi]\,(x \bullet y) = (\kappa(\varphi \circ (x)) \bullet y + x \bullet (\kappa \circ \varphi(y)) - (\varphi \circ \kappa(x)) \bullet y - x \bullet (\varphi \circ \kappa(y)). \tag{8}$$

Due to (7) and (8) we get $x \bullet ([\kappa, \varphi])(y) = ([\kappa, \varphi])(x) \bullet y = 0$.

Let's now $[\kappa, \varphi]$ be a central derivation of $\mathcal{T}$. Then

$$\begin{aligned} \kappa \circ \varphi(x \bullet y) &= [\kappa \circ \varphi]\,(x \bullet y) + (\varphi \circ \kappa)(x \bullet y) \\ &= \varphi(\circ\kappa(x) \bullet y) + \varphi(x \bullet \kappa(y)) \\ &= (\varphi \circ \kappa)(x) \bullet y + x \bullet (\varphi \circ \kappa)(y), \end{aligned}$$

where $\bullet$ represents the products $\dashv, \vdash$ and $\perp$ respectively. □

5.2 Centroids of low-dimensional associative trialgebras

Let $(\mathcal{T}, \dashv, \vdash, \perp)$ be an $n-$dimensional triassocitive algebra with basis $\{e_i\}\,(1 \leq i \leq n)$ and let ϑ be a centroid on $\mathcal{T}$ For any $i, j, k \in \mathbb{N}$, $1 \leq i, j, k \leq n$, let us put

$$e_i \dashv e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k\,;\; e_i \vdash e_j = \sum_{k=1}^{n} \delta_{ij}^k e_k\,;\; e_i \perp e_j = \sum_{k=1}^{n} \xi_{ij}^k e_k, \text{ and } \vartheta(e_i) = \sum_{j=1}^{n} \vartheta_{ji} e_j.$$

Then, in term of basis elements, equation (6) is equivalent to

$$\left\{ \begin{array}{l} \sum_{k=1}^{n}(\gamma_{ij}^k \vartheta_{pk} - \vartheta_{ki}\gamma_{kj}^p) = 0,\; \sum_{k=1}^{n} \gamma_{ij}^k \vartheta_{pk} - \vartheta_{kj}\gamma_{ik}^p) = 0,\; i, j, p \in \{1, \ldots, n\} \\ \sum_{k=1}^{n}(\delta_{rs}^t \vartheta_{qt} - \vartheta_{tr}\delta_{ts}^q) = 0,\; \sum_{k=1}^{n}(\delta_{rs}^t \vartheta_{qt} - \vartheta_{ts}\delta_{rt}^q) = 0,\; r, s, q \in \{1, \ldots, n\} \\ \sum_{k=1}^{n}(\xi_{pq}^r \vartheta_{sr} - \vartheta_{rp}\xi_{rq}^s) = 0,\; \sum_{k=1}^{n}(\xi_{pq}^r \vartheta_{sr} - \vartheta_{rq}\xi_{pr}^s) = 0,\; p, q, s \in \{1, \ldots, n\}\,. \end{array} \right. \tag{9}$$

Theorem 5.8. *The centroids of 2-dimensional complex associative trialgebra are given as follows :*

$$\mathcal{T}_2^1 : \begin{pmatrix} \vartheta_{11} & 0 \\ 0 & \vartheta_{11} \end{pmatrix};\; \mathcal{T}_2^2 : \begin{pmatrix} \vartheta_{22} & 0 \\ 0 & \vartheta_{22} \end{pmatrix};\; \mathcal{T}_2^3 : \begin{pmatrix} \vartheta_{11} & 0 \\ 0 & \vartheta_{11} \end{pmatrix};\; \mathcal{T}_2^4 : \begin{pmatrix} \vartheta_{22} & 0 \\ 0 & \vartheta_{22} \end{pmatrix};$$

$$\mathcal{T}_2^5 : \begin{pmatrix} \vartheta_{11} & 0 \\ 0 & \vartheta_{11} \end{pmatrix};\; \mathcal{T}_2^6 : \begin{pmatrix} \vartheta_{22} & 0 \\ 0 & \vartheta_{22} \end{pmatrix};\; \mathcal{T}_2^7 \begin{pmatrix} \vartheta_{11} & 0 \\ 0 & \vartheta_{11} \end{pmatrix};\; \mathcal{T}_2^8 : \begin{pmatrix} \vartheta_{11} & 0 \\ 0 & \vartheta_{11} \end{pmatrix}.$$

Theorem 5.9. *The centroids of 3-dimensional complex associative trialgebra are given as follows :*

$$\mathcal{T}_3^1 : \begin{pmatrix} \vartheta_{11} & 0 & 0 \\ 0 & \vartheta_{11} & 0 \\ 0 & 0 & \vartheta_{11} \end{pmatrix};\mathcal{T}_3^2 : \begin{pmatrix} \vartheta_{11} & 0 & 0 \\ 0 & \vartheta_{11} & 0 \\ 0 & 0 & \vartheta_{11} \end{pmatrix};\mathcal{T}_3^3 : \begin{pmatrix} \vartheta_{11} & 0 & 0 \\ \vartheta_{21} & \vartheta_{11} & 0 \\ 0 & 0 & \vartheta_{11} \end{pmatrix};$$

$$\mathcal{T}_3^4:\begin{pmatrix}\vartheta_{33}&0&\vartheta_{13}\\0&\vartheta_{33}&0\\0&0&\vartheta_{33}\end{pmatrix};\mathcal{T}_3^5\begin{pmatrix}p_{12}&\vartheta_{13}&\vartheta_{13}\\0&-\vartheta_{32}+\vartheta_{33}&0\\-\vartheta_{32}&\vartheta_{32}&\vartheta_{33}\end{pmatrix};\mathcal{T}_3^6:\begin{pmatrix}\vartheta_{33}&0&0\\\vartheta_{21}&\vartheta_{33}&\vartheta_{23}\\0&0&\vartheta_{33}\end{pmatrix};$$

$$\mathcal{T}_3^7:\begin{pmatrix}\vartheta_{32}+\vartheta_{33}&0&0\\\vartheta_{21}&p_{11}&\vartheta_{23}\\\vartheta_{31}&2\vartheta_{11}-\vartheta_{33}&\vartheta_{33}\end{pmatrix};\mathcal{T}_3^8:\begin{pmatrix}\vartheta_{33}&\vartheta_{12}&\vartheta_{13}\\0&\vartheta_{33}&0\\0&0&\vartheta_{33}\end{pmatrix};\mathcal{T}_3^9:\begin{pmatrix}\vartheta_{33}&0&0\\0&\vartheta_{33}&0\\0&\vartheta_{32}&\vartheta_{33}\end{pmatrix};$$

$$\mathcal{T}_3^{10}:\begin{pmatrix}\vartheta_{11}&0&0\\0&\vartheta_{11}&\vartheta_{23}\\0&0&\vartheta_{11}\end{pmatrix};\mathcal{T}_3^{11}:\begin{pmatrix}\vartheta_{33}&0&0\\0&\vartheta_{33}&0\\\vartheta_{31}&\vartheta_{32}&\vartheta_{33}\end{pmatrix};\mathcal{T}_3^{12}:\begin{pmatrix}\vartheta_{11}&0&0\\0&\vartheta_{11}&\vartheta_{23}\\0&0&\vartheta_{11}\end{pmatrix}.$$

Where $p_{11}=\vartheta_{32}+\vartheta_{33}-\vartheta_{23},\quad p_{12}=-\vartheta_{13}-\vartheta_{32}+\vartheta_{33}$.

Proof. From Theorem 5.9, we provide the proof only for one case to illustrate the approach used, the other cases can be carried out similarly with or no modification(s). Let's consider $\mathcal{T}_3^{12}$. Applying the systems of equations (9). we get $\vartheta_{12}=\vartheta_{13}=\vartheta_{21}=\vartheta_{31}=\vartheta_{32}=0,\quad \vartheta_{22}=\vartheta_{11},\quad \vartheta_{33}=\vartheta_{11}$. Hence, the derivations of $\mathcal{T}_3^{12}$ are given as follows $\vartheta_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\vartheta_2=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$ is basis of $Der(\Gamma)$ and $\mathrm{Dim}Der(\Gamma)=2$. The centroids of the remaining three-dimensional associative trialgebras can be determined in a similar manner as above. □

Theorem 5.10. *The centroids of 4-dimensional associative trialgebra are given as follows :*

$$\mathcal{T}_4^1:\begin{pmatrix}\vartheta_{11}&0&0&0\\\vartheta_{21}&\vartheta_{11}&\vartheta_{23}&0\\0&0&\vartheta_{11}&0\\\vartheta_{41}&0&\vartheta_{43}&\vartheta_{11}\end{pmatrix};\mathcal{T}_4^2\begin{pmatrix}\vartheta_{44}&0&0&0\\\vartheta_{21}&\vartheta_{44}&\vartheta_{23}&0\\0&0&\vartheta_{44}&0\\\vartheta_{41}&0&\vartheta_{43}&\vartheta_{44}\end{pmatrix};\mathcal{T}_4^3:\begin{pmatrix}\vartheta_{44}&0&0&0\\\vartheta_{21}&\vartheta_{44}&\vartheta_{23}&0\\0&0&\vartheta_{44}&0\\\vartheta_{41}&0&\vartheta_{43}&\vartheta_{44}\end{pmatrix};$$

$$\mathcal{T}_4^4:\begin{pmatrix}\vartheta_{22}&0&0&0\\0&\vartheta_{22}&0&0\\0&0&\vartheta_{22}&0\\\vartheta_{41}&\vartheta_{42}&\vartheta_{43}&\vartheta_{22}\end{pmatrix};\mathcal{T}_4^5:\begin{pmatrix}\vartheta_{22}&0&0&0\\0&\vartheta_{22}&0&0\\0&0&\vartheta_{22}&0\\\vartheta_{41}&\vartheta_{42}&\vartheta_{43}&\vartheta_{22}\end{pmatrix};$$

$$\mathcal{T}_4^6:\begin{pmatrix}\vartheta_{44}-\vartheta_{12}&\vartheta_{12}&\vartheta_{13}&\vartheta_{14}\\\vartheta_{44}-\vartheta_{22}&\vartheta_{22}&\vartheta_{23}&\vartheta_{24}\\0&0&\vartheta_{44}&0\\0&0&0&\vartheta_{44}\end{pmatrix};\mathcal{T}_4^7:\begin{pmatrix}\vartheta_{44}-\vartheta_{13}&\vartheta_{12}&\vartheta_{13}&\vartheta_{14}\\0&\vartheta_{44}&0&0\\\vartheta_{44}-\vartheta_{33}&\vartheta_{32}&\vartheta_{33}&\vartheta_{34}\\0&0&0&a_{44}\end{pmatrix};$$

$$\mathcal{T}_4^8:\begin{pmatrix}\vartheta_{44}-\vartheta_{13}&\vartheta_{12}&\vartheta_{13}&\vartheta_{14}\\0&\vartheta_{44}&0&0\\\vartheta_{44}-\vartheta_{33}&\vartheta_{32}&\vartheta_{33}&\vartheta_{34}\\0&0&0&a_{44}\end{pmatrix};\mathcal{T}_4^9:\begin{pmatrix}\vartheta_{44}&\vartheta_{12}&0&\vartheta_{14}\\0&\vartheta_{44}&0&0\\0&\vartheta_{32}&\vartheta_{44}&\vartheta_{34}\\0&0&0&\vartheta_{44}\end{pmatrix}$$

$$\mathcal{T}_4^{10}:\begin{pmatrix}\vartheta_{44}&\vartheta_{12}&0&\vartheta_{14}\\0&\vartheta_{44}&0&0\\0&\vartheta_{32}&\vartheta_{44}&\vartheta_{34}\\0&0&0&\vartheta_{44}\end{pmatrix};\mathcal{T}_4^{11}:\begin{pmatrix}\vartheta_{44}-\vartheta_{13}&\vartheta_{12}&\vartheta_{13}&\vartheta_{14}\\0&\vartheta_{44}&0&0\\\vartheta_{44}-\vartheta_{33}&\vartheta_{32}&\vartheta_{33}&\vartheta_{34}\\0&0&0&\vartheta_{44}\end{pmatrix};$$

$$;\mathcal{T}_4^{12}:\begin{pmatrix}\vartheta_{44}-\vartheta_{13} & \vartheta_{12} & \vartheta_{13} & \vartheta_{14}\\ 0 & \vartheta_{44} & 0 & 0\\ \vartheta_{44}-\vartheta_{33} & \vartheta_{32} & \vartheta_{33} & \vartheta_{34}\\ 0 & 0 & 0 & \vartheta_{44}\end{pmatrix};\mathcal{T}_4^{13}:\begin{pmatrix}\vartheta_{44} & 0 & 0 & 0\\ 0 & \vartheta_{44} & 0 & 0\\ 0 & 0 & \vartheta_{44} & 0\\ \vartheta_{41} & \vartheta_{42} & \vartheta_{43} & \vartheta_{44}\end{pmatrix};$$

$$\mathcal{T}_4^{14}:\begin{pmatrix}\vartheta_{44} & \vartheta_{12} & 0 & \vartheta_{14}\\ 0 & \vartheta_{44} & 0 & 0\\ 0 & \vartheta_{32} & \vartheta_{44} & \vartheta_{34}\\ 0 & 0 & 0 & \vartheta_{44}\end{pmatrix};\mathcal{T}_4^{15}:\begin{pmatrix}\vartheta_{44} & 0 & 0 & 0\\ 0 & \vartheta_{44} & 0 & 0\\ \vartheta_{31} & \vartheta_{32} & \vartheta_{44} & \vartheta_{34}\\ 0 & 0 & 0 & \vartheta_{44}\end{pmatrix};\mathcal{T}_4^{16}:\begin{pmatrix}\vartheta_{33} & 0 & 0 & 0\\ \vartheta_{21} & \vartheta_{33} & \vartheta_{23} & 0\\ 0 & 0 & \vartheta_{33} & 0\\ \vartheta_{41} & 0 & \vartheta_{43} & \vartheta_{33}\end{pmatrix}.$$

Proof. From Theorem 5.9, we provide the proof only for one case to illustrate the approach used, the other cases can be carried out similarly with or no modification(s). Let's consider $\mathcal{T}_4^4$. Applying the systems of equations (9). we get $\vartheta_{12}=\vartheta_{13}=\vartheta_{14}=\vartheta_{21}=\vartheta_{23}=\vartheta_{24}=\vartheta_{31}=\vartheta_{32}=\vartheta_{43}=0$, $\vartheta_{11}=\vartheta_{22}$,
$\vartheta_{33}=\vartheta_{22}\quad \vartheta_{44}=\vartheta_{22}$. Hence, the derivations of $\mathcal{T}_4^4$ are given as follows

$$\vartheta_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\vartheta_2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}\ \vartheta_3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{pmatrix},\vartheta_4=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$$

is basis of $Der(\Gamma)$ and $\mathrm{Dim}Der(\Gamma)=4$. The centroid of the remaining parts of four-dimensional associative trialgebras can be carried out in a similar manner as shown above. □

Corollary 5.11.

- *The dimensions of the centroids of two-dimensional associative trialgebras are one.*
- *The dimensions of the centroids of three-dimensional associative trialgebras range between one and five.*
- *The dimensions of the centroids of four-dimensional associative trialgebras range between one and seven.*

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