**Galois J. Algebra** **1** (2025), no. 1, pp. 43-50.

Galois Journal of Algebra

https://gja.intrasci.net/

# A family of rank 4 non-algebraic matroids with pseudomodular dual

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**ABSTRACT**

The Tic-Tac-Toe matroid is a paving matroid of rank 5 on 9 elements which is pseudomodular and whose dual is non-algebraic. It has been proposed as a possible example of an algebraic matroid whose dual is not algebraic. We present an infinite family of matroids sharing these properties and generalizing the Tic-Tac-Toe matroid.

**ARTICLE HISTORY**

Received: 23 October 2025
Revised: 26 November 2025
Accepted: 27 November 2025
Published: 10 December 2025
Communicated by
Anda Olteanu
https://doi.org/10.65908/gja.2025.24976

**KEYWORDS**

algebraic matroids;
pseudomodular lattices.

**MSC**

05B35;
12F20.

## 1 Introduction

Hassler Whitney [20] introduced matroids in 1935 as structures that capture the "abstract properties of linear dependence". In the second edition of his "Moderne Algebra" van der Waerden [19] remarked in 1937, that algebraic dependencies in field extensions share these abstract properties, a fact that had already been shown by Steinitz in 1910 [18].

**Definition 1.1.** *Let $k$ be a field and $k \subseteq K$ a field extension. A finite subset $\{v_1, \ldots, v_k\} \subseteq K$ is called* algebraically dependent *over $k$ if there exists a non-trivial polynomial $0 \neq p(x_1, \ldots, x_k) \in k[x_1, \ldots, x_k]$ such that*

$$p(v_1, \ldots, v_k) = 0.$$

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*The set $K$ is called* algebraically independent*, otherwise.*

Algebraic independence satisfies the axioms of matroid theory (see e.g. 6.7.1 in [17]).

**Definition 1.2.** *A matroid $M = (E, \mathcal{I})$ on a finite set $E$ is called* algebraic*, if there exists a field extension $k \subseteq K$ a subset $E' \subseteq K$ and a bijection $\sigma : E \to E'$ such that for all $I \subseteq E$*

$$I \in \mathcal{I} \Leftrightarrow \sigma(I) \text{ is algebraically independent over } k.$$

It has been shown by Ingleton [11] only in 1971 that algebraic matroids are a strict superclass of the class of linear matroids and it took until 1975 for Ingleton and Main [12] to show that "Non-algebraic matroids exist", by proving that the Vámos matroid is non-algebraic. For that purpose they showed that in an algebraic matroid the three bounding lines of a prism must intersect in a common point of the closure of the representing field. This key lemma was generalized by Andreas Dress and Laszlo Lovász to the "series reduction theorem" [8]. We will use it to give an exact proof that the Vámos matroid is non-algebraic in the next section.

Björner and Lovász [6] then used the "series reduction theorem" to define pseudomodular lattices.

Considering the question of whether pseudomodularity of the geometric lattice of a matroid is sufficient for the existence of a lattice theoretical dual, an adjoint of that matroid, Marion Alfter and Winfried Hochstättler [2] presented a pseudomodular matroid of rank 5 which does not admit an adjoint and named it the Tic-Tac-Toe matroid. It will appear in Section 4 as $M_3^*$. It had been observed independently by Marion Alfter[1] and by Bernt Lindström [15] that the dual of this matroid is non-algebraic, using an application of the key lemma of Ingleton and Main.

Algebraic matroids are closed under taking minors and under truncation. It is still an open problem though, whether they are closed under matroid duality.

Since pseudomodularity in a certain sense seems to capture the most important properties of algebraic matroids, it was suggested by Laszlo Lovász [14] that the Tic-Tac-Toe matroid might be a good candidate as a counterexample for the unsolved question whether the dual of an algebraic matroid is algebraic again.

We tried to popularize this suggestion with a preprint [9] in 1997 and a note in the Proceedings of the Graph Theory V [10]. We were partially successful. The author is aware of at least two PhD projects which were motivated by the question, whether the Tic-Tac-Toe matroid is algebraic or not, namely by Stephan Kromberg [13] and Guus Bollen [7]. Although both theses present very nice results, they did not settle the original question.

The paper is organized as follows. In the next section we present the combinatorial properties of algebraic matroids that led to the definition of pseudomodularity. In Section 3 we define an infinite family of rank 4 paving matroids generalizing the dual of the Tic-Tac-Toe matroid and prove that all of them are non-algebraic. In Section 4 we show that all their duals are pseudomodular, and we end with some remarks and open questions in Section 5. We assume some familiarity with matroid theory, the standard reference is [17].

## 2 Combinatorial Properties of Algebraic Matroids

**Definition 2.1.** *Let $M = (E, \mathcal{I})$ be a matroid on a finite set $E$ and $S \subseteq A \subseteq E$. Then $S$ is* in series in $A$ *if contracting $A \setminus S$ turns $S$ into a circuit.*

**Theorem 2.2** (Dress-Lovász 1987)**.** *Let $M = (E, \mathcal{I})$ be an algebraic matroid represented by a set $E' \subseteq K$ over a field $k$, where $K$ is algebraically closed.*

*Let $S' \subseteq E'$ be in series in $A' \subseteq E'$. Then there exists $\beta \in K$ such that $\forall T' \subseteq A' \setminus S'$:*

$$S' \cup T' \text{ is algebraically dependent} \Leftrightarrow \beta \cup T' \text{ is algebraically dependent.}$$

Thus, in an algebraic matroid every series admits a shortcut. From this it is seen that the Vámos matroid is non-algebraic as follows:

The Vámos matroid $V$ is a paving matroid, i.e. all its circuits have $\text{rank}(V)$ or $\text{rank}(V)+1$ elements, on an eight point set $\{a, a', b, b', c, c', d, d'\}$. Its five 4-element circuits or *circuit hyperplanes* are given by $\{a, a', b, b'\}$, $\{a, a', c, c'\}$, $\{b, b', c, c'\}$, $\{b, b', d, d'\}$ and $\{c, c', d, d'\}$. In particular $\{a, a', d, d'\}$ is independent. In the following we denote the closure of a set $T \subseteq V$ in $K$, i. e. the set of elements of $K$ algebraically dependent from $T$, by $\text{cl}(T)$. Assume $V$ were algebraically represented over a field extension $k \subseteq K$. Since $\{a, a'\}$ is in series in $\{a, a', b, b', c, c'\}$ there exists $\beta_1$ in the algebraic closure of $K$ which lies on the intersection of the two "lines" $\beta_1 \in \text{cl}(\{b, b'\}) \cap \text{cl}(\{c, c'\})$. We also have $\beta_1 \in \text{cl}(\{a, a', b, b'\}) \cap \text{cl}(\{a, a', c, c'\}) = \text{cl}(\{a, a'\})$. Hence, according to the series reduction theorem the three bounding lines of the "prism" $a, a', b, b', c, c'$ must intersect in a point in the algebraic closure of the representing field.

By symmetry, there also exists $\beta_2 \in \text{cl}(\{a, a'\}) \cap \text{cl}(\{b, b'\}) \cap \text{cl}(\{d, d'\})$. Since $\beta_1$ and $\beta_2$ lie in the intersection of the same lines they must be parallel. Thus the two lines $\text{cl}(\{a, a'\})$ and $\text{cl}(\{d, d'\})$ have an intersection of rank at least one, contradicting the independence of the set $\{a, a', d, d'\}$.

All classical proofs of non-algebraicity, known to the author, apply the series reduction theorem, postulate additional points and derive a contradiction. Another method, which is based on non-Shannon inequalities, has recently been presented in [3]. It uses the fact that algebraic matroids are non-isotropic [16].

## 3 The Primal Matroids

For ease of definition we start with the generalizations of the dual of the Tic-Tac-Toe Matroid which are all non algebraic.

**Definition 3.1.** *Let $k \geq 3$ and $E = \{a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_k\}$. We define a paving matroid (see [17], Proposition 1.3.10) $M_k$ of rank 4 on $E$ by defining a set of circuit hyperplanes. These are given by*

$$\{\{a_i, a_j, b_i, b_j\} \mid i < j\} \cup \{\{a_i, a_j, c_i, c_j\} \mid i < j, (i, j) \neq (1, k)\} \cup \{\{b_i, b_j, c_i, c_j\} \mid i \neq j\}.$$

*Note that $\{a_1, a_k, c_1, c_k\}$ is a basis and not a circuit hyperplane.*

In Figure 1 the circuit hyperplanes of $M_k$ are all four point sets that form a rectangle except for the four corner points $\{a_1, a_k, c_1, c_k\}$.

$$\begin{array}{ccccccc} a_1 & - & a_2 & - & \dots & - & a_k \\ | & & | & & | & & | \\ b_1 & - & b_2 & - & \dots & - & b_k \\ | & & | & & | & & | \\ c_1 & - & c_2 & - & \dots & - & c_k \end{array}$$

Figure 1: A graphical guide to $M_k$

**Theorem 3.2.** *$M_k$ is not algebraic.*

*Proof.* $M_k$ may be considered as a complete graph, where each edge has been replaced by a prism, but one hyperplane $\{a_1, a_k, c_1, c_k\}$ of the prism given by the edge $\{1, k\}$ is broken. In particular the four point flats given by the edge $\{k-1, k\}$, namely $\{a_{k-1}, a_k, b_{k-1}, b_k\}$, $\{a_{k-1}, a_k, c_{k-1}, c_k\}$ and $\{b_{k-1}, b_k, c_{k-1}, c_k\}$, as well as those for $\{1, k-1\}$, namely $\{a_1, a_{k-1}, b_1, b_{k-1}\}$, $\{a_1, a_{k-1}, c_1, c_{k-1}\}$ and $\{b_1, b_{k-1}, c_1, c_{k-1}\}$ form such a prism. Note that the "prism" of the edge $\{1, k\}$ is the one with the broken hyperplane (see Figure 2 left for a visualization of $M_3$).

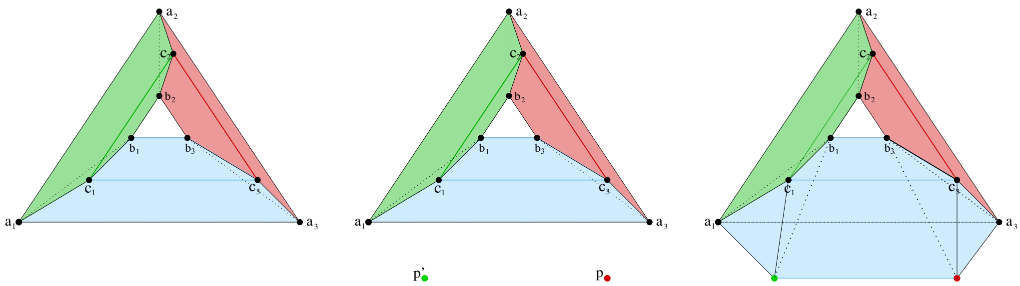

Figure 2: $M_3$ is non-algebraic

Assume $M_k$ were algebraic. According to the Ingleton-Main Lemma (compare the proof of non-algebraicity of the Vámos-matroid) there exist

$$p \in \mathrm{cl}(\{a_{k-1}, a_k\}) \cap \mathrm{cl}(\{b_{k-1}, b_k\}) \cap \mathrm{cl}(\{c_{k-1}, c_k\}) \text{ and}$$
$$p' \in \mathrm{cl}(\{a_1, a_{k-1}\}) \cap \mathrm{cl}(\{b_1, b_{k-1}\}) \cap \mathrm{cl}(\{c_1, c_{k-1}\}).$$

These are indicated by the green resp. the red dot in the middle of Figure 2. Hence $\{p, p'\}$ is a subset of $\mathrm{cl}(\{a_{k-1}, a_k, a_1\})$ as well as of $\mathrm{cl}(\{b_{k-1}, b_k, b_1\})$ and $\mathrm{cl}(\{c_{k-1}, c_k, c_1\})$. This implies that the eight points $\{a_1, b_1, c_1, a_k, b_k, c_k, p, p'\}$ form a Vamos matroid (indicated in light blue in Figure 2 to the right), contradicting algebraic representability. □

**Remark 3.3.** *The choice of the third vertex $k-1$ in the triangle is arbitrary. It could be replaced by any vertex from 2 to $k-1$ to give rise to a copy of $M_3$, the dual of the Tic-Tac-Toe matroid.*

## 4 The Dual Matroids

Since $M_k$ is sparse paving, so is its dual $M_k^*$. Note, that its circuit hyperplanes are given by the complements of the circuit hyperplanes of $M_k$.

We will show that $M_k^*$ is pseudomodular for all $k \geq 3$. Since we have to deal with both matroid theoretic as well as lattice theoretical concepts we will frequently identify elements of the geometric lattice with the set of elements of the matroid in the corresponding flat.

**Definition 4.1.** *Let $M$ be a matroid and $L$ be its geometric lattice of closed flats. Then $M$ is* pseudomodular *[6], if $\forall a, b, c \in L$ :*

$$\begin{aligned} & r(a \vee b \vee c) - r(a \vee b) = r(a \vee c) - r(a) = r(b \vee c) - r(b) \qquad (1) \\ \Longrightarrow \quad & r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a). \qquad (2) \end{aligned}$$

**Theorem 4.2.** *$M_k^*$ is pseudomodular.*

*Proof.* Assume $M_k^*$ were not pseudomodular and let $a, b, c \in L$ violate Definition 4.1. Since $r(a \vee c) + r(a \vee b) = r(a \vee b \vee c) + r(a)$ the flats $a \vee c$ and $a \vee b$ must form a modular pair with meet $a$. By symmetry we also have $(b \vee c) \wedge (a \vee b) = b$ and hence

$$(a \vee c) \wedge (b \vee c) \wedge (a \vee b) = a \wedge b. \qquad (3)$$

By submodularity we have

$$\begin{aligned} & r((a \vee c) \wedge (b \vee c)) + r(a \vee b) \\ \geq \quad & r((a \vee c) \wedge (b \vee c)) \vee (a \vee b)) + r((a \vee c) \wedge (b \vee c) \wedge (a \vee b)) \\ = \quad & r(a \vee b \vee c) + r(a \wedge b). \end{aligned}$$

Thus, violating Definition 4.1 implies

$$r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) \geq r(a \vee b \vee c) - r(a \vee b) + 1 = r(a \vee c) - r(a) + 1. \qquad (4)$$

By definition of $M_k^*$ any flat $F$ with $r(F) \leq r(M_k^*) - 2$ is independent, implying $r(F) = |F|$. Now

$$\begin{aligned} |a \vee c| \quad & = \quad |(a \vee c) \wedge (a \vee b)| + |(a \vee c) \wedge (b \vee c)| - |(a \vee c) \wedge (b \vee c) \wedge (a \vee b)| \\ & \overset{3}{=} \quad r((a \vee c) \wedge (a \vee b)) + r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) \\ & \overset{4}{\geq} \quad r((a \vee c) \wedge (a \vee b)) + r(a \vee c) - r(a) + 1 \\ & \geq \quad r(a) + r(a \vee c) - r(a) + 1 \\ & = \quad r(a \vee c) + 1. \end{aligned}$$

Hence, $(a \vee c)$ must be a circuit hyperplane of $M_k^*$ and by symmetry this also holds for

$(b \vee c)$. A similar computation shows that the same is true for $a \vee b$:

$$\begin{aligned}
|a \vee b| &= r((a \vee b) \wedge (b \vee c)) + r((a \vee b) \wedge (a \vee c)) - r(a \wedge b) \\
&= r(b) + r(a) - r(a \wedge b) \\
&\overset{4}{\geq} r(b) + r(a \vee c) - r((a \vee c) \wedge (b \vee c)) + 1 \\
&\overset{\text{subm.}}{\geq} r(b) + r(a \vee c) - (r((a \vee c) + r(b \vee c)) - r(a \vee b \vee c)) + 1 \\
&= r(b) - r(b \vee c) + r(a \vee b \vee c) + 1 \\
&\overset{1}{=} r(a \vee b) + 1.
\end{aligned}$$

Using the defining properties of $a, b, c$ this moreover implies

$$r(a) = r(b) = r(a \vee b \vee c) - 2.$$

Now, if $r((a \vee c) \wedge (b \vee c)) \leq r(a \vee b \vee c) - 3$ we conclude $r(a \wedge b) \leq r(a \vee b \vee c) - 5$ and thus

$$\begin{aligned}
|a \vee b| &= r(a) + r(b) - r(a \wedge b) \\
&\overset{4}{\geq} r(a) + r(b)) - r(a \vee b \vee c) + 5 \\
&= r(a \vee b \vee c)) + 1,
\end{aligned}$$

contradicting $|a \vee b| = r(a \vee b \vee c)$ as $a \vee b$ is a circuit hyperplane.

Hence, $a \vee b, a \vee c$ and $b \vee c$ are three circuit hyperplanes which pairwise intersect in the three different colines $a, b$ and $(a \vee c) \wedge (b \vee c)$. But say $E \setminus \{a_i, a_j, b_i, b_j\}$ intersects only two other circuit hyperplanes, namely $E \setminus \{a_i, a_j, c_i, c_j\}$ and $E \setminus \{b_i, b_j, c_i, c_j\}$ (provided $(i, j) \neq (1, k)$), in the coline $E \setminus \{a_i, a_j, b_i, b_j, c_i, c_j\}$. Hence, the three circuit hyperplanes could pairwise intersect only in the same coline, a contradiction.

We conclude that $M_k^*$ must be pseudomodular. □

## 5 Remarks and Open Problems

Stephan Kromberg [13] analyzes two matroids which differ only little from the Tic-Tac-Toe matroid, one of them linear, the other one non-algebraic.

Michael Bamiloshin et al. [4] recently enumerated all rank 4 matroids on 8 elements that are not linearly representable and found 3 examples, where they could not decide whether they are algebraic or not. They furthermore described several paving matroids of rank 5 which are related to the Tic-Tac-Toe Matroid in [5] and refer to [7] concerning their representability status.

It is not difficult to construct a pseudomodular matroid which contracts to a non-algebraic matroid, and thus is non-algebraic itself. It might be worthwhile to examine whether some of our matroids $M_k^*$ are non-algebraic for some $k \geq 4$.

The non-algebraicity of the matroids $M_k$ is not very exciting, as all of them contain $M_3$. We wonder whether our construction of replacing the edges of a graph by prisms could avoid that, i.e. does there exist a triangle-free two-connected graph, such that replacing its edges by prisms and relaxing (breaking) one circuit hyperplane yields a non-algebraic matroid?

**Acknowledgement**

The author is grateful to an anonymous referee for suggestions improving the presentation of the paper.

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