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Some Perspectives on Distribution of strongly persistent clutters

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ABSTRACT

This survey aims to present and promote the study of the strong persistence property in clutters from a quantitative and structural perspective. We denote by $\mathfrak{C}_n$ the set of all clutters on $[n]$, by $\mathfrak{S}_n$ the subset of clutters satisfying the SPP, and by $\mathfrak{N}_n = \mathfrak{C}_n \setminus \mathfrak{S}_n$ those that fail it. Our analysis focuses on understanding how these families behave and interact across consecutive levels.

Beyond results, this paper seeks to draw attention to open problems and conjectures related to the structure and distribution of persistent clutters, proposing a new research direction based on the study of families $\mathfrak{S}_n$ and $\mathfrak{N}_n$ across levels of n.

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1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring. A monomial ideal I is *squarefree* if $G(I)$ consists of squarefree monomials. A *clutter* is a pair $\mathcal{C} = (V, E)$ where V is a finite set and E is a set of subsets of V such that if $a \subseteq b$ with $a, b \in E$, then $a = b$. The sets $V = V(\mathcal{C})$

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and $E = E(\mathcal{C})$ are called vertex set and edge set, respectively. If $f = \{x_{i_1}, \dots, x_{i_r}\} \in E(\mathcal{C})$, then denote by $\tilde{f}$ the squarefree monomial $x_{i_1} \cdots x_{i_r}$. The *edge ideal* of the clutter $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal generated by $\{\tilde{f} \mid f \in E(\mathcal{C})\}$. This assignment defines a natural bijection between squarefree monomial ideals of $K[x_1, \dots, x_n]$ and clutters whose vertex set is X.

We say that an ideal I of a commutative ring R has the *strong persistence property* if

$$I^{k+1} : I = I^k \quad \text{for every } k \geq 1.$$

In the case of monomial ideals, this property can be studied directly through the behavior of their monomial generators. Indeed, if $G(I) = \{m_1, \dots, m_r\}$ denotes the minimal monomial generating set of I, it is enough to verify the condition for the monomials in $R := k[x_1, \cdots, x_n]$ the *polynomial ring*. More precisely, I satisfies the strong persistence property provided that for any monomial $m \in \mathrm{Mon}(R)$ (set of all monomials of R) satisfying

$$m m_i = \ell_i \, m_1^{a_{i1}} \cdots m_r^{a_{ir}}, \qquad \text{with } \sum_{j=1}^{r} a_{ij} = k+1, \; i \in [r],$$

there exists a monomial ℓ such that

$$m = \ell \, m_1^{b_{i1}} \cdots m_r^{b_{ir}}, \qquad \text{where } \sum_{j=1}^{r} b_{ij} = k.$$

Hence, strong persistence property can be verified through these multiplicative relations among the minimal monomial generators of I.

This formulation is particularly useful when studying edge ideals of clutters. Since every generator of $I(\mathcal{C})$ corresponds to an edge of the clutter $\mathcal{C}$, the above multiplicative relations among monomials translate directly into combinatorial conditions on the intersections of edges. Consequently, the strong persistence property for $I(\mathcal{C})$ can be understood in purely combinatorial terms, allowing us to analyze it through the structure of $\mathcal{C}$ itself rather than through the algebraic behavior of its associated primes. This viewpoint establishes a bridge between the algebraic properties of monomial ideals and combinatorial configurations within the clutter, which is one of the main motivations of this study.

If $\mathcal{C}$ is a clutter over $X = \{x_1, \dots, x_n\}$, we say that $\mathcal{C}$ is *strongly persistent* when its edge ideal $I(\mathcal{C})$ satisfies the strong persistence property. Empirical evidence suggests that only a very small proportion of clutter fails to satisfy SPP (Strong persistence Property). Although the absolute number of non-persistent clutters increases with n, their proportion relative to the total number of clutters remains extremely low.

We denote by $\mathfrak{C}_n$ the set of all clutters on $[n]$, $\mathfrak{S}_n$ the set of clutters on $[n]$ satisfying SPP and $\mathfrak{N}_n = \mathfrak{C}_n \setminus \mathfrak{S}_n$. The comparative study of these quantities across consecutive levels provides valuable insight into the distribution of persistent clutters and their asymptotic behavior. In particular, it is natural to ask whether the limits

$$\lim_{n \to \infty} \frac{|\mathfrak{N}_n|}{|\mathfrak{S}_n|}, \qquad \lim_{n \to \infty} \frac{|\mathfrak{N}_n|}{|\mathfrak{C}_n|}$$

exist, moreover, if these limits take the value 0. A vanishing limit would suggest that non-persistence becomes increasingly rare as the number of vertices grows, indicating that persistence might be an asymptotically dominant or stable property among large clutters.

The main goal of this work is to develop a combinatorial and algebraic framework to estimate, compare, and bound the values of $|\mathfrak{S}_n|$, $|\mathfrak{N}_n|$, and $|\mathfrak{C}_n|$, while understanding how the strong persistence property behaves under structural operations on clutters. By analyzing both ascending operations (such as cones, edge cones, and corona products) and descending ones (such as deletions and contractions), we aim to obtain recursive relations or inequalities connecting these quantities across levels $n-1$, n, and $n+1$.

This approach not only clarifies the mechanisms that preserve persistence but also provides a systematic way to approximate the relative density of strongly persistent clutters. Ultimately, our objective is to describe the asymptotic distribution of clutters with SPP and to establish quantitative bounds that may lead to a deeper understanding of persistence as a structural invariant in combinatorial commutative algebra.

Through a combination of constructive lemmas and propositions involving cones, suspensions, and contractions, it is shown that all clutters on five vertices are strongly persistent. This conclusion supports the conjecture that strong persistence is an asymptotically dominant and structurally stable phenomenon in the universe of clutters, revealing deep connections between combinatorial configurations and algebraic invariants in monomial ideal theory.

In [21] and [2], we developed algebraic and combinatorial techniques to analyze how $|\mathfrak{N}_n|$ changes under vertex addition and deletion. More precisely, we would like to find adequate functions $k(n), K(n) > 0$ such that

$$k(n)\,|\mathfrak{C}_{n-1}| \leq |\mathfrak{N}_n| \leq K(n)\,|\mathfrak{C}_{n+1}| \quad \text{for every } n.$$

And

$$k(n)\,|\mathfrak{S}_{n-1}| \leq |\mathfrak{N}_n| \leq K(n)\,|\mathfrak{S}_{n+1}| \quad \text{for every } n.$$

These inequalities allow us to control the growth of $\mathfrak{N}_n$ between consecutive levels and to relate the behavior of non-persistent clutters at different scales.

2 Set of monomials and monomial Ideals

Definition 2.1. *Let $\mathfrak{M}$ be a set of monomials. A monomial $m \in \mathfrak{M}$ is said to be* minimal *if for every $m' \in \mathfrak{M}$ such that $m' \mid m$, it follows that $m' = m$. Similarly, a monomial $m \in \mathfrak{M}$ is* maximal *if for every $m' \in \mathfrak{M}$ such that $m \mid m'$, we have $m' = m$. We denote by $\mathfrak{M}_{\min}$ and $\mathfrak{M}_{\max}$ the sets of minimal and maximal elements of $\mathfrak{M}$, respectively.*

Given $A \subseteq [n]$ and $a \in \mathbb{Z}^n_{\geq 0}$, we define the restriction

$$a_{|A} := (a_i)_{i \in A}.$$

For $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, we write

$$x^a := x_1^{a_1} \cdots x_n^{a_n} \in K[x_1, \dots, x_n].$$

Given $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{Z}^n_{\geq 0}$, we note that $x^a \mid x^b$ if and only if $a \leq b$, that is, $a_i \leq b_i$ for all $i \in [n]$. Additionally, we denote $a < b$ if and only if $a \leq b$ and $a \neq b$

lemma 2.2. *Let $\mathfrak{M}$ be an infinite set of monomials in $R = K[x_1, \dots, x_n]$. Then there exists a sequence $\{m_i\}_{i \in \mathbb{N}} \subseteq \mathfrak{M}$ such that $m_i \mid m_{i+1}$ and $a_i \neq a_{i+1}$ for all i.*

Proof. We consider

$$M = \{a \in \mathbb{Z}^n_{\geq 0} \mid x^a \in \mathfrak{M}\}.$$

Its sufficient to show that there exists a sequence $a_i \in M$, $i \in \mathbb{N}$ such that $a_i < a_{i+1}$

Given $\emptyset \subsetneq A \subseteq [n]$ and $b \in \mathbb{Z}^A_{\geq 0}$, we define

$$M_A^b := \{a \in M \mid a_{|A} = b\} \quad \text{and} \quad N_A^b := \{a_{|A^c} \mid a \in M_A^b\}.$$

Where $A^c = [n] \backslash A$. Clearly, $|N_A^b| = |M_A^b|$ since $a \longmapsto a_{|A^c}$ define a bijection.

If N_A^b is infinite for some $A \subseteq [n]$ with $1 \leq |A| \leq n-1$ and $b \in \mathbb{Z}^A_{\geq 0}$. By the induction hypothesis, there exists a sequence $a_i \in M$, $i \in \mathbb{N}$ such that $a_i' := (a_i)_{|A^c}$ holds

$$a_i' < a_{i+1}', \; i \in \mathbb{N}$$

Since $(a_i)_{|A} = b$ for each $i \in \mathbb{N}$, we obtain $a_i < a_{i+1}$.

Now, suppose N_A^b is finite for every $A \subseteq [n]$ with $1 \leq |A| \leq n-1$ and every $b \in \mathbb{Z}^A_{\geq 0}$. Choose

$$a_1 = (a_{11}, \dots, a_{1n})$$

to be a minimal element of $\mathfrak{M}$. For each $j \in [n]$ and $0 \leq b \leq a_{1j}$ we have that $N_{\{j\}}^{(b)}$ is finite and there are

$$\sum_{j \in [n]} (a_{1j} + 1),$$

sets of kind $N_{\{i\}}^{(b)}, j \in [n], 0 \leq b \leq a_{1j}$. Hence both

$$\bigcup_{j \in [n],\, 0 \leq b \leq a_{1j}} N_{\{j\}}^{(b)} \quad \text{and} \quad \bigcup_{j \in [n],\, 0 \leq b \leq a_{1ij}} M_{\{j\}}^{(b)}$$

are finite.

Define

$$M_1 := M \setminus \left(\bigcup_{j \in [n],\, 0 \leq b \leq a_{1j}} M_{\{j\}}^{(b)} \right).$$

We obtain $a_1 \notin M_1$ and M_1 is infinite. More over, if $a \in M_1$. Thus $a_1 < a$. Hence we choose $a_2 \in M_1$. By continuing this process recursively, we obtain a strictly increasing sequence

$$a_1 < a_2 < a_3 < \cdots.$$

Finally,

$$x^{a_1} \mid x^{a_2} \mid x^{a_3} \mid \cdots$$

is a sequence in $\mathfrak{m}$ with $x^{a_i} \neq x^{a_{i+1}}$ for every i. □

This proposition immediately yields a generalization of Dickson's Lemma. Indeed, by applying the construction above to any infinite set of monomials in $R = K[x_1, \ldots, x_n]$, we can extract an infinite chain under divisibility, showing that every set of monomials admits only finitely many minimal elements. Hence, the classical Dickson's Lemma follows as a direct consequence of the finiteness of $\mathfrak{M}_{\min}$ established above.

Corollary 2.3 (Dickson's Lemma)**.** *Let $\mathfrak{M}$ be a set of monomials. Then the set $\mathfrak{M}_{\min}$ of minimal elements is finite.*

Recall that a monomial ideal $I \subset R = K[x_1, \ldots, x_n]$ has dimension 0 if and only if it is primary to the irrelevant maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. Indeed, $\dim(R/I) = 0$ means that every variable x_i is nilpotent modulo I, hence some power $x_i^{k_i} \in I$ for each i, implying that $\sqrt{I} = \mathfrak{m}$. Conversely, if $\sqrt{I} = \mathfrak{m}$, then all variables become nilpotent in R/I, so $\dim(R/I) = 0$. Thus, zero-dimensional monomial ideals and ideals primary to the irrelevant maximal ideal are equivalent notions.

The following results extend the classical statement of Dickson's Lemma from the setting of monomials to that of monomial ideals. In the monomial case, Dickson's Lemma asserts that every set of monomials in a polynomial ring contains only finitely many minimal elements under divisibility. Theorems 2.4 and Corollary 2.6 generalize this finiteness property to collections of monomial ideals, showing that any infinite family of such ideals must contain two members comparable under inclusion. In this sense, the lattice of monomial ideals inherits the same well-quasi-order structure that the monomials themselves possess under divisibility.

lemma 2.4. *[14, Lemma 5.1] Let $\mathcal{I}$ be an infinite collection of zero-dimensional monomial ideals in a polynomial ring $R = K[x_1, \ldots, x_n]$. Then there exist two ideals $I, J \in \mathcal{I}$ such that $I \subseteq J$.*

Proof. Assume by contradiction, that $\mathcal{I}$ consist of infinitely many zero-dimensional monomial ideals which are pairwise incomparable under inclusion.

Select an ideal $I_1 \in \mathcal{I}$. Since for each $I \in \mathcal{I} \setminus \{I_1\}$ it holds that $I \nsubseteq I_1$, every such I must include at least one of the finitely many standard monomials of I_1. Consequently, there exists an infinite subcollection of ideals in $\mathcal{I}$ that share the same subset of standard monomials of I_1. Denote this subcollection by $\mathcal{I}_1$ and let J_1 be the intersection of all ideals in $\mathcal{I}_1$.

We proceed inductively. Suppose that $\mathcal{I}_k$ and J_k have already been defined. Choose an ideal $I_{k+1} \in \mathcal{I}_k$. As in the previous step, one can find an infinite subcollection $\mathcal{I}_{k+1} \subseteq \mathcal{I}_k$ consisting of ideals having the same nontrivial intersection with the set of standard monomials of I_{k+1}. Define J_{k+1} as the intersection of all ideals in $\mathcal{I}_{k+1}$.

By construction, we have $J_{k+1} \supseteq J_k$, since J_{k+1} contains all standard monomials that are common to the ideals in $\mathcal{I}_{k+1}$. Repeating this procedure, we obtain an infinite strictly increasing chain

$$J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \cdots$$

of monomial ideals in S, contradicting the Noetherian property of S. Hence the elements of $\mathcal{I}$ are comparable. That is, there exist $I, J \in \mathcal{I}$ with $I \neq J$ such that $I \subseteq J$.

□

Theorem 2.5. *[14, Theorem 1.1] Let $\mathcal{I}$ be an infinite collection of monomial ideals in a polynomial ring over an arbitrary field. Then there exist two ideals $I, J \in \mathcal{I}$ such that $I \subseteq J$.*

Proof. Every monomial ideal of R has only finitely many associated primes, all of which are monomial primes of the form

$$P_\tau = (x_i \mid i \notin \tau), \qquad \tau \subseteq [n].$$

Hence, among the ideals in $\mathcal{I}$ we may restrict to an infinite subfamily sharing the same set of associated primes; denote this subfamily again by $\mathcal{I}$.

For each $I \in \mathcal{I}$, fix an irredundant primary decomposition

$$I = \bigcap_\tau I_\tau,$$

where each I_τ is a monomial ideal primary to P_τ. If for a given τ the set $\{I_\tau : I \in \mathcal{I}\}$ is finite, then infinitely many ideals in $\mathcal{I}$ have the same τ–component I_τ. Otherwise, by applying Lemma 2.2 to the ring $K[x_i : i \notin \tau]$, we obtain an infinite descending chain

$$I_\tau^{(1)} \supsetneq I_\tau^{(2)} \supsetneq I_\tau^{(3)} \supsetneq \cdots .$$

Since there are only finitely many possible associated primes, we can select an infinite sequence of ideals

$$I_1, I_2, I_3, \ldots \in \mathcal{I}$$

such that for each fixed τ, the corresponding primary components satisfy

$$I_{1,\tau} \supseteq I_{2,\tau} \supseteq I_{3,\tau} \supseteq \cdots .$$

Consequently, the intersections

$$I_k = \bigcap_\tau I_{k,\tau}$$

form an infinite descending chain

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots .$$

In particular, we have $I_{k+1} \subseteq I_k$ for some k, showing that two ideals in $\mathcal{I}$ are comparable under inclusion.

This contradicts the assumption that $\mathcal{I}$ is an infinite antichain. Hence, any infinite collection of monomial ideals in R contain ideals $I \neq J$ such that $I \subseteq J$. □

Corollary 2.6. *[14, Corollary 5.2] Let $\mathcal{I}$ be an infinite collection of zero-dimensional monomial ideals. Then there exists an infinite strictly descending chain*

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$$

of ideals in $\mathcal{I}$.

Proof. Because the ring R is Noetherian, the collection $\mathcal{I}$ contains ideals that are maximal with respect to inclusion. By Lemma 2.4, there are only finitely many such maximal elements. Select one of them, say $I_1 \in \mathcal{I}$, which contains infinitely many other ideals from $\mathcal{I}$.

Set

$$\mathcal{I}_1 = \{\, I \in \mathcal{I} \mid I \subsetneq I_1 \,\}.$$

Applying the same argument recursively to $\mathcal{I}_1$, we obtain an infinite strictly descending sequence of ideals

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots,$$

where each I_{k+1} is properly contained in I_k. This establishes the desired chain and completes the proof. □

As a consequence of the previous results, we can recover a well-known corollary concerning initial ideals. Indeed, since any infinite collection of monomial ideals must contain two ideals that are comparable under inclusion, it follows that only finitely many distinct initial ideals can arise when all monomial orders are considered. This classical finiteness statement—often attributed to Galligo and Bayer–Stillman—emerges naturally from the general framework developed above, showing that the space of initial ideals is itself finite.

Corollary 2.7. *Let I be an ideal of the polynomial ring $R = K[x_1, \ldots, x_n]$. Then the set*

$$\{\, \mathrm{in}_{\le}(I) \ : \le \in \Omega \,\}$$

of initial ideals (as $\le$ ranges over all monomial orders Ω on R) is finite.

Proof. Fix a monomial order $\le$ on R. By [7, Proposition. 2.2.5], the residue classes of the monomials not contained in $\mathrm{in}_{\le}(I)$ form a k-basis of R/I; denote this set of standard monomials by

$$B_{\le}(I) := \{\, [m] \in R/I \mid m \notin \mathrm{in}_{\le}(I) \,\}.$$

If $\le$ and $\le'$ are monomial orders with $\mathrm{in}_{\le}(I) \subseteq \mathrm{in}_{\le'}(I)$, then every monomial that is standard for $\le'$ is also standard for $\le$, hence

$$B_{\le'}(I) \subseteq B_{\le}(I).$$

But both $B_{\le'}(I)$ and $B_{\le}(I)$ are k-bases of R/I, so the inclusion forces equality:

$$B_{\le'}(I) = B_{\le}(I) \quad \text{and hence} \quad \mathrm{in}_{\le'}(I) = \mathrm{in}_{\le}(I).$$

Therefore distinct initial ideals are pairwise incomparable under inclusion; that is, the family $\{\mathrm{in}_{\le}(I)\}_{\le \in \Omega}$ is an antichain in the poset of monomial ideals. By Theorem 2.4, every such antichain is finite, and the claim follows. □

Having established that such families are necessarily finite, the next natural step is to determine explicit bounds for their cardinalities. We are particularly interested in finding estimates that depend only on the number of variables (or equivalently, on the number of vertices n in the combinatorial interpretation). In other words, once finiteness is guaranteed by the previous results, our goal is to quantify this finiteness by providing asymptotic or exact bounds that reflect the combinatorial complexity of the ambient space.

Given a finite set X and E consisting of subsets of X whose elements do not have contention relations, E is called a clutter, nonetheless originally a family that was called a Sperner family. The *Lubell–Yamamoto–Meshalkin (LYM) inequality* is a cornerstone of extremal set theory, providing a probabilistic bound on the structure of families of subsets within the Boolean lattice. Formally, it asserts that for any family $\mathcal{F} \subseteq 2^{[n]}$, the inequality

$$\sum_{A\in\mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1$$

holds whenever $\mathcal{F}$ contains no full chain. This result generalizes Sperner's theorem and has become a unifying tool in the study of antichains, posets, and related combinatorial structures. Beyond its original formulation, the LYM inequality has inspired numerous extensions to distributive lattices, weighted versions, and applications in information theory and extremal combinatorics, making it a fundamental reference point in survey expositions of the field.

lemma 2.8. *[13, Theorem 1] Let $\mathcal{C}$ be a clutter. We consider s_k the number of edges of $E(\mathcal{C})$ of cardinality k, for $k = 1, \ldots, n$. Then*

$$\sum_{k=1}^{n} \frac{s_k}{\binom{n}{k}} \leq 1$$

Proof. Given $e \in E$ with $|e| = k$, we consider

$$B_e := \{f : [k] \longrightarrow e \mid f \text{ is a bijection}\}.$$

And

$$B'_e := \{g : [n]\backslash[k] \longrightarrow V\backslash e \mid g \text{ is a bijection}\}.$$

If $A_e \subseteq S_n$ denotes the set of all permutations $\sigma \in S_n$ such that $\sigma([k]) = e$, then

$$|A_e| = |B_e|\,|B'_e| = k!(n-k)!.$$

Since $B_e \cap B_{e'} = \emptyset$ for distinct $e, e' \in E$, we have that

$$A := \bigcup_{e\in E} A_e \subseteq S_n$$

is a pairwise disjoint union of subsets of S_n. Hence,

$$\sum_{k=1}^{n} s_k\, k!(n-k)! = \sum_{e\in E} |e|!(n-|e|)! = |A| \leq |S_n| = n!.$$

Therefore,

$$\sum_{k=1}^{n} s_k\, k!(n-k)! \leq n!.$$

Dividing by $n!$ and using the identity $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we obtain

$$\sum_{k=1}^{n} \frac{s_k}{\binom{n}{k}} = \sum_{k=1}^{n} s_k \frac{k!(n-k)!}{n!} \leq 1.$$

□

Theorem 2.9. *[22, Theorem 1] Let $\mathcal{C} = (V, E)$ be a clutter on a finite set V of cardinality n. Then*

$$|E| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. Since $\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all $k = 1, \ldots, n$, it follows from Proposition 2.8 that

$$\sum_{k=1}^{n} \frac{s_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{k=1}^{n} \frac{s_k}{\binom{n}{k}} \leq 1.$$

Hence,

$$|E| = \sum_{k=1}^{n} s_k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

□

3 Squarefree monomial ideals and clutters

The following lemma provides a useful combinatorial tool for studying the strong persistence property in clutters, particularly for low powers of their edge ideals. It allows one to control divisibility relations among squarefree monomials appearing in consecutive powers, ensuring that certain containment relations between colon ideals hold. In practice, this technical observation simplifies the verification of strong persistence for the first symbolic or ordinary powers of a clutter ideal and serves as a key step in proving the next two results.

lemma 3.1. *[21, Lemma 1] Let f, g be squarefree monomials in $R = K[x_1, \ldots, x_n]$. If there exists an integer $k \geq 2$ and a monomial m such that $f^k \mid mg$, then $f^{k-1} \mid m$.*

Corollary 3.2. *[21, Corollary 4] Let I be a squarefree monomial ideal. If the minimal generating set $G(I)$ has at most two elements, then I satisfies the strong persistence property.*

Theorem 3.3. *[21, Theorem 6] Let I be a squarefree monomial ideal in $R = K[x_1, \ldots, x_n]$. Then*

$$(I^2 : I) = I.$$

These results play a fundamental role in understanding the behavior of the strong persistence property. In particular, Corollary 3.2 and Theorem 3.3 provide the base cases for analyzing the equality $(I^k : I) = I^{k-1}$ when $k \geq 3$. Once this equality is verified for $k = 2$ and for ideals with a small number of generators, the next challenge is to study how the property extends to higher powers and to ideals generated by more than two squarefree monomials. This transition marks the point where the combinatorial structure of the clutter associated to I becomes essential.

Theorem 3.4. *[21, Theorem 7] Let $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r$ be a clutter with connected components $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Then $\mathcal{C}$ has the strong persistence property if and only if some of its components $\mathcal{C}_i$ has the strong persistence property.*

Theorem 3.4 provides a structural simplification in the study of the strong persistence property. It shows that it is enough for a single connected component of a clutter to satisfy the property in order for the entire clutter to have it. Consequently, one may restrict the analysis to connected clutters, since any failure or validity of the property is determined locally by the connected components. Moreover, the theorem also implies that the property fails if and only if each connected component fails, which is a conceptually natural behavior in mathematics. Considering the low probability that a clutter fails to satisfy strong persistence (as will be discussed later), the existence of such exceptional case is particularly interesting. Identifying and classifying the minimal clutters that fail the property, and understanding how they embed within larger nonpersistent clutters, becomes an essential direction for further investigation.

Example 3.5. *[21, Example 2] Let $\mathcal{C}$ be a clutter. If $e_1, e_2 \in \{A \subseteq V(\mathcal{C}) \mid A \cap f = \varnothing$ for every $f \in E(\mathcal{C})\}$, then by Theorem 3.4 and Corollary 3.2, the clutter $E(\mathcal{C}) \cup \{e_1, e_2\}$ satisfies the strong persistence property.*

This example plays a fundamental illustrative role in our framework since it provides a constructive way to move upward in the hierarchy of clutters, that is, from the family $\mathfrak{C}_n$ of all clutters on n vertices to the family $\mathfrak{S}_m$ of strongly persistent clutters with $m > n$. Indeed, by adding suitable disjoint edges to a given clutter, we can ensure the strong persistence property holds in the resulting structure. Moreover, as will be shown later in Proposition 4.10, every clutter with exactly three edges satisfies the strong persistence property. Hence, this idea naturally extends the previous example: if $\mathcal{C}$ is a clutter and $e_1, e_2, e_3 \subseteq V(\mathcal{C})$ such that elements of $\{e_1, e_2, e_3\}$ are incomparable and $e_i \cap e = \varnothing$ for every $e \in E(\mathcal{C})\}$ and $i \in [3]$. Then the clutter whose edge set is $E(\mathcal{C}) \cup \{e_1, e_2, e_3\}$ satisfies the strong persistence property.

lemma 3.6. *[21, Lemma 2] Let $\mathcal{C}$ be a clutter. If there exists an edge $e_0 \in E(\mathcal{C})$ such that*

$$A = \{\, e \cap e_0 \mid e \in E(\mathcal{C}) \,\}$$

forms a chain under inclusion, then the edge ideal $I(\mathcal{C})$ satisfies the strong persistence property.

While Lemma 3.6 may appear to be a rather simple and highly particular case. However, when one considers its contrapositive form, the statement reveals a deeper connection with several well-studied algebraic properties, such as the König property and the Cohen–Macaulay condition. This perspective highlights an underlying structural link between the strong persistence property and broader concepts in the theory of Gorenstein rings, thereby opening a pathway between persistence phenomena in clutters and classical results in combinatorial commutative algebra.

Corollary 3.7. *[21, Corollary 6] If $\mathcal{C}$ is a clutter that does not satisfy the strong persistence property, then for every $e \in E(\mathcal{C})$ there exist edges $e_1, e_2 \in E(\mathcal{C})$ such that*

$$e \cap e_1 \nsubseteq e \cap e_2 \quad \text{and} \quad e \cap e_2 \nsubseteq e \cap e_1.$$

Corollary 3.7 can be viewed as the negative counterpart of Lemma 3.6. While Lemma 3.6 identifies a structural condition that guarantees the strong persistence property, this corollary characterizes, in combinatorial terms, the configurations that force its failure. The result provides a natural bridge toward the study of algebraic properties such as the Cohen–Macaulay and König conditions. Before establishing this connection, we briefly recall several key concepts that will allow us to interpret the strong persistence property in a more algebraic and homological framework.

Definition 3.8. *Let $\mathcal{C}$ be a clutter. A subset $A \subseteq V(\mathcal{C})$ is called a* vertex cover *if $A \cap e \neq \varnothing$ for every $e \in E(\mathcal{C})$. The* cover number *of a clutter $\mathcal{C}$ is defined as*

$$\alpha_0(\mathcal{C}) = \min\{\, |A| \mid A \text{ is a vertex cover of } \mathcal{C}\,\}.$$

Definition 3.9. *A clutter $\mathcal{C}$ is said to be* unmixed *if every minimal vertex cover B satisfies $|B| = \alpha_0(\mathcal{C})$.*

Definition 3.10. *A* matching *of a clutter $\mathcal{C}$ is a set of pairwise disjoint edges $\{e_1, \dots, e_s\}$. A matching $\{e_1, \dots, e_s\}$ of $\mathcal{C}$ is called* perfect *if $\bigcup_{i=1}^{s} e_i = V(\mathcal{C})$.*

Definition 3.11. *A clutter $\mathcal{C}$ is called a* König clutter *if there exists a matching with exactly $\alpha_0(\mathcal{C})$ edges.*

Proposition 3.12 ([16], Theorem 4.9)**.** *Let $\mathcal{C}$ be a König clutter. Then $\mathcal{C}$ is unmixed if and only if there exists a perfect matching $e_1, \dots, e_g$ with $g = \alpha_0(\mathcal{C})$ such that the following condition holds: for any two distinct edges $e, e' \in E(\mathcal{C})$ and any two distinct vertices $x \in e$ and $y \in e'$ contained in some e_i, one has that*

$$(e \setminus \{x\}) \cup (e' \setminus \{y\})$$

contains an edge of $\mathcal{C}$.

This result plays a central role in connecting the combinatorial structure of König clutters with algebraic properties such as unmixedness and the Cohen–Macaulay condition. In particular, it can be interpreted as the structural counterpart of the negative form of Lemma 3.6 while that lemma identifies situations guaranteeing the strong persistence property, this characterization describes the combinatorial obstructions that prevent it. The interplay between these two viewpoints reveals that the failure of strong persistence is intimately related to the failure of the unmixedness condition, thereby highlighting a deep link between persistence phenomena and the classical theory of König and Cohen–Macaulay clutters.

Definition 3.13. *The* incidence matrix *of a clutter* $\mathcal{C}$*, denoted by* $A_{\mathcal{C}}$*, is the matrix whose columns are the characteristic vectors of the edges of* $\mathcal{C}$*. An* r-cycle *of* $\mathcal{C}$ *is an alternant sequence* $x_{i_1}, e_{j_1} \dots x_{i_r}, e_{j_r}$ *of vertices and edges such that its correspond square submatrix* $r \times r$ *submatrix of* $A_{\mathcal{C}}$ *having exactly two entries equal to* 1 *in each row and in each column.*

The following theorem provides the anticipated bridge connecting the strong persistence property with other fundamental algebraic conditions, such as the Cohen–Macaulay and König properties. It shows that under suitable combinatorial restrictions specifically, the absence of 4-cycles in a König unmixed clutter automatically satisfies strong persistence. This result establishes a direct link between persistence phenomena and the homological behavior of the associated monomial ring, highlighting that strong persistence can be interpreted as a combinatorial manifestation of Cohen–Macaulayness in this setting.

Theorem 3.14. *[21, Theorem 8] Let* $\mathcal{C}$ *be a König unmixed clutter. If* $\mathcal{C}$ *does not contain any* 4*-cycle, then* $\mathcal{C}$ *satisfies the strong persistence property.*

Example 3.15. *[15, Example 2.18] Let* $\mathcal{C}_0$ *be the clutter with vertex set* $\{x_1, \dots, x_6\}$ *and edge set*

$$E(\mathcal{C}_0) = \{\, x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_5,\ x_1x_4x_6,\ x_1x_5x_6,$$

$$x_2x_3x_6,\ x_2x_4x_5,\ x_2x_5x_6,\ x_3x_4x_5,\ x_3x_4x_6 \,\}.$$

Since

$$(I(\mathcal{C}_0)^3 : I(\mathcal{C}_0)) \neq I(\mathcal{C}_0)^2,$$

the clutter $\mathcal{C}_0$ *does not satisfy the strong persistence property.*

The clutter $\mathcal{C}_0$, originally mentioned in this context by Villarreal, has become a classical and somewhat puzzling example in the study of the persistence properties of monomial ideals. It fails to satisfy the strong persistence property (SPP), nevertheless have the persistence property (PP). This makes $\mathcal{C}_0$ a striking counterexample in the ongoing effort to characterize persistence in terms of purely combinatorial invariants.

4 Cones and contractions

Since one of our main goals is to compare the number of clutters on $[n]$ that satisfy the strong persistence property with those defined on $[n-1]$ and $[n+1]$, we introduce a sequence of propositions that allow us to locate, bound, and relate these quantities across different levels. These results provide a combinatorial framework for understanding how the persistence behavior evolves when expanding or contracting the ground set, that is, when moving upward or downward between consecutive levels in the hierarchy of clutters.

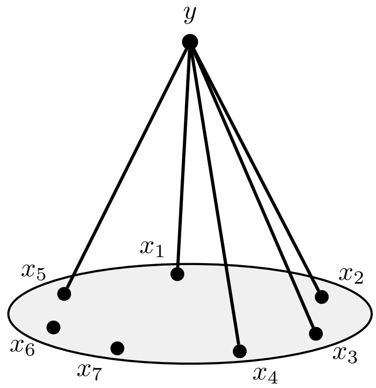
Definition 4.1. *Let* $X = \{x_1, \dots, x_r\}$ *be a finite set,* $A \subseteq X$ *and let* $y \notin X$*. We define the* simple cone *as the clutter*

$$\mathcal{A}^y = (X \cup \{y\}, E),$$

where $E = \{X\} \cup \{\, \{y, x_i\} \mid x_i \in A \,\}$.

Example 4.2. *Consider* $\mathcal{A}^y$ *where* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $A = \{x_1, x_2, x_3, x_4, x_5\}$

$$E(\mathcal{A}^y) = \{X, \{y, x_1\}, \{y, x_2\}, \{y, x_3\}, \{y, x_4\}, \{y, x_5\}\}$$



Proposition 4.3. *[21, Proposition 9] Let* X *be a set,* $A \subseteq X$ *and* $x \notin X$*. Then* $\mathcal{A}^x$ *has the strong persistence property.*

Definition 4.4. *Let* $\mathcal{C} = (X, E)$ *be a clutter with* $X = \{x_1, \ldots, x_r\}$ *a finite set and* $y \notin X$*. We define the* cone *of* $\mathcal{C}$ *as the clutter*

$$\mathcal{C}y = (X \cup \{y\}, E_y),$$

where $E_y = \{\, e \cup \{y\} \mid e \in E \,\}$.

Theorem 4.5. *([21, Proposition 6]) Let* $\mathcal{C}$ *be a clutter. Then* $\mathcal{C}$ *satisfies the persistence property if and only if its cone* $\mathcal{C}y$ *also satisfies the persistence property.*

Theorem 4.6. *([21, Proposition 5]) Let* $\mathcal{C}$ *be a clutter. Then* $\mathcal{C}$ *satisfies the strong persistence property if and only if its cone* $\mathcal{C}y$ *also satisfies the strong persistence property.*

This theorem shows that the persistence and strong persistence properties are stable under the cone operation, a fundamental type of expansion that preserves the combinatorial structure while increasing the dimension. Conceptually, it means that the persistence behavior of a clutter depends on intrinsic relations among its edges rather than on the specific number of vertices. Consequently, the cone construction provides a convenient tool for inductive arguments on the size of the vertex set and for building higher-dimensional examples from known persistent clutters.

Definition 4.7. *Let* $\mathcal{C} = (X, E)$ *be a clutter with* $X = \{x_1, \ldots, x_r\}$ *a finite set and* $y \notin X$*. We define the* edge cone *of* $\mathcal{C}$ *as the clutter*

$$\mathcal{C}_y = (X \cup \{y\}, E_y),$$

where $E_y = E \cup \{\, \{y, x_i\} \mid x_i \in X \,\}$.

Conjecture 4.8. *Let* $\mathcal{C}$ *be a clutter. Then* $\mathcal{C}$ *satisfies the (strong) persistence property if and only if its edge cone* $\mathcal{C}_y$ *satisfies the (strong) persistence property.*

The conjecture extends the previous theorem to the edge cone, a more intricate construction that combines both the original edges of $\mathcal{C}$ and the new edges incident to the added vertex y. If true, this statement would imply that the persistence behavior of clutters is invariant under one of the most natural combinatorial extensions, bridging the gap between local and global stability. In particular, proving this conjecture would strengthen the connection between persistence properties and structural operations on clutters, providing a unified framework for understanding how algebraic invariants evolve under controlled expansions. From a broader perspective, it would suggest that persistence is not only a homological invariant but also a combinatorially stable feature within the class of monomial ideals associated to clutters.

Proposition 4.9. *[21, Proposition 7] $\mathcal{C} = (V, E)$ has the strong persistence property if and only if $\mathcal{C}' = (V, E')$ has the strong persistence property, where $E' = \{e \setminus \bigcap_{e' \in E} e' \mid e \in E\}$.*

This result provides an alternative way to interpret the strong persistence property through the operation of intersecting edges. In particular, if the intersection of all edges in $\mathcal{C}$ is nonempty, we can view the clutter as a sequence of cones obtained by successively adding each element of this intersection as a new vertex shared by all edges. Hence, the persistence behavior of $\mathcal{C}$ can be studied inductively by analyzing the corresponding cones. This perspective not only simplifies the combinatorial structure but also clarifies the geometric intuition behind the preservation of the strong persistence property under cone-like extensions.

Proposition 4.10. *[21, Proposition 8] A clutter $\mathcal{C}$ with 3 edges has the strong persistence property.*

Theorem 4.11. *[21, Theorem 9] If I is a squarefree monomial ideal in $K[x_1, x_2, x_3, x_4]$, then I has the strong persistence property.*

Corollary 4.12. *[21, Corollary 7] If $I \subseteq K[x_1, \ldots, x_n]$ is a squarefree monomial ideal without the strong persistence property, then $n \geq 5$ and there is $k \geq 3$ such that $(I^k : I) \neq I^{k-1}$.*

Corollary 4.13. *Let $\mathcal{C} = (X, E)$ be a clutter, let $x \notin X$, and fix $x_i \in X$. Define*

$$\mathcal{C}'_{x_i} = \big(X \cup \{x\},\, E'\big) \qquad \text{with} \qquad E' = E \cup \big\{\{x, x_i\}\big\}.$$

Then $I(\mathcal{C}'_{x_i})$ has the strong persistence property.

Proof. Set $e_0 = \{x, x_i\} \in E'$. Consider

$$A \;=\; \{\, e \cap e_0 \mid e \in E' \,\}.$$

If $e \in E$, then $e \cap e_0$ is either $\emptyset$ or $\{x_i\}$ depending on whether $x_i \in e$; for $e = e_0$ we have $e \cap e_0 = e_0 = \{x, x_i\}$. Hence

$$A = \{\emptyset, \{x_i\}, \{x, x_i\}\},$$

which is a chain under inclusion. By Lemma 3.6, this implies that $I(\mathcal{C}')$ has the strong persistence property. □

This construction provides a simple lifting device: starting from any clutter on X, by adjoining a new vertex x and a single edge $\{x, x_1\}$ we obtain a clutter on $X \cup \{x\}$ that satisfies strong persistence. Operationally, it converts arbitrary clutters on a given level into persistent clutters one level up, and will be useful to build families of examples and to compare the distribution of clutters with (strong) persistence across adjacent vertex sets.

Definition 4.14. *Let $\mathcal{C} = (V, E)$ be a clutter with $x \in V$. The* contraction *of x is the clutter $\mathcal{C}/x$ with vertex set $V \setminus \{x\}$ and edge set* $\min\{e \setminus \{x\} \mid e \in E\}$.

Definition 4.15. *Let $\mathcal{C} = (V, E)$ be a clutter and let $x \in V$ be a vertex. The* deletion *(or* vertex deletion*) of $\mathcal{C}$ with respect to x is the clutter*

$$\mathcal{C} \setminus x = (V \setminus \{x\}, E'),$$

where

$$E' = \{\, e \in E \mid x \notin e \,\}.$$

That is, $\mathcal{C} \setminus x$ is obtained from $\mathcal{C}$ by removing the vertex x and all edges that contain x.

Example 4.16. *[21, Example 4] Let $\mathcal{C}_0'$ be the clutter with vertex set $V(\mathcal{C}_0) \cup \{x\}$ with $x \notin V(\mathcal{C}_0)$ and edge set*

$$E(\mathcal{C}_0') = E(\mathcal{C}_0) \cup \{\{x, x_1\}\},$$

where $\mathcal{C}_0$ is as in Corollary 4.13 above. By that corollary, $I(\mathcal{C}_0')$ has the strong persistence property, while $\mathcal{C}_0' \setminus x = \mathcal{C}_0$ does not.

Moreover, we can remove any vertex from $\mathcal{C}_0'$ and still obtain a clutter whose edge ideal satisfies the strong persistence property. Indeed, if we delete any vertex different from x and x_1, the resulting clutter still contains the edge $\{x, x_1\}$, and therefore, by applying the same reasoning as in Corollary 4.13, it retains the property. On the other hand, if we remove x_1, the resulting clutter has five vertices, and as we shall see later, all clutters with five vertices satisfy the strong persistence property. This observation highlights the robustness of the construction: the addition of the edge $\{x, x_1\}$ not only induces persistence in the extended clutter but also stabilizes it under vertex deletions.

Example 4.17. *Let $\mathcal{C}_0$ be the Villarreal clutter introduced in [15, Example 3.2], which does not satisfy the strong persistence property. $\mathcal{C}_0'$ has the strong persistence property. However $(\mathcal{C}_0')_{\setminus x}$, with $x \notin V(\mathcal{C}_0)$ does have strong persistence property and the clutter $\mathcal{C}_0' \setminus x_i$ does have the strong persistence property for each $x_i \in V(\mathcal{C}_0)$.*

This phenomenon is remarkable: while $\mathcal{C}_0$ provides a counterexample to the strong persistence property, all of its vertex deletions restore the property. This suggests that persistence can be more stable in lower-dimensional or smaller vertex configurations. From a combinatorial viewpoint, this behavior is highly informative, as it indicates that studying the persistence of deletions of a clutter may reveal structural mechanisms responsible for the failure of persistence at the original level. In particular, this observation may serve as a useful tool for analyzing persistence in clutters on $n - 1$ vertices and could provide an inductive approach to understanding how the property behaves across consecutive levels in the hierarchy of clutters.

Proposition 4.18. *[21, Proposition 10] Let $\mathcal{C}$ be a clutter and $x \in V(\mathcal{C})$. If $\mathcal{C}$ has the (strong) persistence property, then $\mathcal{C}/x$ has the (strong) persistence property.*

Proposition 4.18 is particularly significant because they allow us to study the persistence and strong persistence properties because it allows us moving *downward* in the vertex hierarchy, that is, from a clutter on n vertices to one on $n-1$ vertices. Together with the results that describe how to extend clutters upward (from n to $n+1$ vertices) while preserving persistence, these propositions provide two complementary approaches to compare and estimate the number of clutters with the SPP at consecutive levels. Hence, we can attempt to bound this quantity both from below and from above, using estimates depending on the levels $n-1$ and $n+1$, respectively. This perspective opens the way to recursive or inductive methods for understanding the distribution of persistent clutters across the vertex hierarchy.

The converse of Proposition 4.18 does not hold. Indeed, let $\mathcal{C}_0$ be the Villarreal clutter described above. For every vertex $x_i \in V(\mathcal{C}_0)$, the contraction $\mathcal{C}_0/x_i$ is a simple graph, and thus its edge ideal satisfies the strong persistence property by [15, Lemma 2.12].

However, the original clutter $\mathcal{C}_0$ does not satisfy the SPP. This fact is especially important, since it shows that a clutter can have all its contractions satisfying the strong persistence property (and analogously the persistence property) while failing to possess the property itself. Such examples where $\mathcal{C}$ fails SPP but every $\mathcal{C}/x_i$ satisfies it is crucial for understanding how persistence behaves under vertex contractions and highlights subtle asymmetries between local and global persistence behavior.

Proposition 4.19. *Let $\mathcal{C} = (X, E)$ be a clutter, and let $\phi : X \longrightarrow Y$ be an injective map. Then the clutter*

$$\phi(\mathcal{C}) = (Y, \phi(E)), \quad \text{where } \phi(E) = \{\phi(e) \mid e \in E\},$$

has the (strong) persistence property if and only if $\mathcal{C}$ has the (strong) persistence property.

Proof. It follows from the fact that ϕ induces a ring isomorphism of $K[X]$ and $K[\phi(X)]$ sending $I(\mathcal{C})$ onto $I(\phi(\mathcal{C}))$. The restriction of this automorphism yields an isomorphism between the two ideals, and hence the (strong) persistence property is preserved.

□

5 Mengerian property and strong persistence property

To deepen the study of the strong persistence property, it is often useful to consider a weaker version that still captures its essential behavior. This leads to the concept of the symbolic strong persistence property, which we introduce next.

Definition 5.1. *Let I be an ideal of a commutative ring R. We say that I satisfies the* symbolic strong persistence property *if $I^{(k+1)} : I^{(1)} = I^{(k)}$ for all $k \geq 1$.*

In [21, Theorem 11] it is proved that the strong persistence property implies the symbolic strong persistence property, and in [12] it is also shown that every square-free monomial ideal satisfies the symbolic strong persistence property. However, the converse statement is false, even within the class of monomial ideals.

In this section, we focus on the study of the strong persistence property in the case of square-free monomial ideals. Before proceeding further, we recall some basic notions and fix notation.

It is well known there is a one-to-one correspondence between the family of square-free monomial ideals of the polynomial ring $K[x_1, \dots, x_n]$ and the class of clutters on the vertex set $\{x_1, \dots, x_n\}$. This correspondence is given by the *edge ideal* of a clutter, denoted by $I(\mathcal{C})$. By a slight abuse of notation, we will use the same symbol to denote both an edge and its corresponding monomial.

Definition 5.2. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and let*

$$f = \sum_{a \in \mathbb{N}^n} c_a x^a \qquad (c_a \in K)$$

be a polynomial. The support *of f is the finite set*

$$\operatorname{supp}(f) := \{x^a \mid c_a \neq 0\}.$$

lemma 5.3. *Let I be a monomial ideal. The following conditions are equivalent:*

(1) I is primary.

(2) If $x_i \mid m$ for some $m \in G(I)$, then there exists an integer $k \geq 1$ such that $x_i^k \in G(I)$.

Proof. (1) $\Rightarrow$ (2): Let $m = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_t}^{\alpha_{i_t}} \in G(I)$, and suppose that $x_{i_j} \mid m$ for some $j \in [t]$. Since $m \in G(I)$, we have $m' = m/x_{i_j}^{\alpha_{i_j}} \notin I$. As I is primary, there exists $r \geq 1$ such that $(x_{i_j}^{\alpha_{i_j}})^r \in I$, hence $x_{i_j}^k \in G(I)$ for some k.

(2) $\Rightarrow$ (1): Let

$$A = \{\, x_i \mid x_i^s \in G(I) \text{ for some } s \,\} \quad \text{and} \quad A' = \{x_1, \dots, x_n\} \setminus A.$$

Take $f, g \in R$ such that $fg \in I$ but $f \notin I$. Assume without loss of generality $\operatorname{supp}(f) \cap I = \emptyset$, since otherwise we could write $f = f_1 + f_2$ with $\operatorname{supp}(f_1) \cap I = \emptyset$ and $\operatorname{supp}(f_2) \subseteq I$, obtaining $f_1 g = fg - f_2 g \in I$ and $f_1 \in I$.

If $g \notin \sqrt{I}$, then there exists $u \in \operatorname{supp}(g)$ such that $x_i \nmid u$ for every $x_i \in A$. Fix a lexicographic monomial order $\leq$ such that $x_i \leq x_j$ for each $x_i \in A'$ and $x_j \in A$. And let

$$v = \min_{\leq}\{\, u \in \operatorname{supp}(g) \mid u \notin \sqrt{I} \,\}.$$

Let $m = \min_{\leq} \operatorname{supp}(f)$. Since $mu \in \operatorname{supp}(fg)$ and $fg \in I$, there exists $h \in G(I)$ such that $h \mid vm$. Because $\gcd(h, v) = 1$, we have $h \mid m$, contradicting $f \notin I$. Hence $g \in \sqrt{I}$, so $g^k \in I$ for some k, and therefore I is primary. $\square$

Corollary 5.4. *If Q is a primary monomial ideal, then Q^k is also a primary ideal for every integer $k \geq 1$.*

Proposition 5.5. *[18, Lemma 5.1] Let $Q_1, \dots, Q_r$ be primary monomial ideals such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$, and suppose that the set $\{\sqrt{Q_1}, \dots, \sqrt{Q_r}\}$ has no containment relations among its elements. If each Q_i satisfies the strong persistence property, then for every $k \geq 1$ one has*

$$\bigcap_{i=1}^{r} Q_i^{k+1} : \bigcap_{i=1}^{r} Q_i = \bigcap_{i=1}^{r} Q_i^{k}.$$

Proof. Let $m \in \bigcap_{i=1}^{r} Q_i^{k+1} : \bigcap_{i=1}^{r} Q_i$. Fix an index $1 \leq j \leq r$ and choose, for each $i \neq j$, an element $x_i \in Q_i \setminus P_j$, where $P_j = \sqrt{Q_j}$. For any $a \in Q_j$, we have

$$max_1 \cdots x_{j-1}x_{j+1} \cdots x_r \in \bigcap_{i=1}^{r} Q_i^{k+1} \subseteq Q_j^{k+1}.$$

Since Q_j is a monomial primary ideal, Q_j^{k+1} is also primary; hence $ma \in Q_j^{k+1}$, and therefore $m \in Q_j^{k+1} : Q_j$. By the strong persistence property, we conclude $m \in Q_j^k$. Thus,

$$m \in \bigcap_{i=1}^{r} Q_i^{k}.$$

The reverse inclusion is immediate, and the proof follows. □

lemma 5.6. *[12, Proposition 5] Every prime monomial ideal satisfies the strong persistence property.*

Proof. Let $P = (x_{i_1}, \dots, x_{i_r})$ be a prime monomial ideal. Consider $m \in P^{k+1} : P$. In particular, $mx_{i_1} \in P^{k+1}$, which means that there exists a monomial m' that is the product of $k+1$ generators of P such that $mx_{i_1} = \ell m'$ for some monomial ℓ. Dividing both sides by x_{i_1} yields $m \in P^k$. Hence, P satisfies the strong persistence property. □

Theorem 5.7. *[18, Theorem 5.1] Every square-free monomial ideal satisfies the symbolic strong persistence property.*

Proof. Let I be a square-free monomial ideal and let $P_1, \dots, P_r$ denote its minimal prime ideals. By Proposition 5.5 and Lemma 5.6, we obtain

$$\bigcap_{i=1}^{r} P_i^{k+1} : \bigcap_{i=1}^{r} P_i = \bigcap_{i=1}^{r} P_i^{k} \quad \text{for every } k.$$

Moreover, by [7, Proposition 1.4.4], we know that

$$I^{(k)} = \bigcap_{i=1}^{r} P_i^k \quad \text{for all } k.$$

Hence,

$$I^{(k+1)} : I = I^{(k)} \quad \text{for every } k,$$

which proves that I satisfies the symbolic strong persistence property. □

Corollary 5.8. *[18, Corollary 5.1] If I is a normally torsion-free square-free monomial ideal, then I satisfies the strong persistence property.*

Proof. By Theorem 5.7, we have

$$I^{(k+1)} : I^{(1)} = I^{(k)} \quad \text{for every } k.$$

Moreover, by [23, Theorem 14.3.6] (v) and (vi), it follows that

$$I^{(k)} = I^k \quad \text{for all } k.$$

Hence, I has the strong persistence property. □

Definition 5.9. *Let $\mathcal{C} = (V, E)$ be a clutter with $V = \{x_1, \ldots, x_n\}$. For a vector $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$, the* parallelization *of $\mathcal{C}$ with respect to w, denoted $\mathcal{C}^w$, is the clutter obtained as follows:*

1. *Replace each vertex x_i by w_i copies*

$$x_i^{(1)}, \ldots, x_i^{(w_i)}.$$

2. *For each edge $e \in E$, form all edges of the form*

$$\{\, x_i^{(j)} \mid x_i \in e,\ 1 \leq j \leq w_i \,\}.$$

The resulting clutter $\mathcal{C}^w$ is called the parallelization *of $\mathcal{C}$.*

Theorem 5.10. *[23, Theorem 14.3.6] Let $\mathcal{C} = (V, E)$ be a clutter and let $I(\mathcal{C}) \subset K[x_1, \ldots, x_n]$ be its edge ideal. The following statements are equivalent.*

1. *Every parallelization $\mathcal{C}^w$ of $\mathcal{C}$ satisfies the König property;*

$$\beta_1(\mathcal{C}^w) = \alpha_0(\mathcal{C}^w) \qquad \textit{for all } w \in \mathbb{N}^V.$$

2. $I(\mathcal{C})$ *is* normally torsion free*; that means,*

$$\mathrm{Ass}(I(\mathcal{C})^k) \subseteq \mathrm{Ass}(I(\mathcal{C})) \qquad \textit{for all } k \geq 1.$$

3. The symbolic and ordinary powers of $I(\mathcal{C})$ *coincide:*

$$I(\mathcal{C})^{(k)} = I(\mathcal{C})^k \qquad \textit{for all } k \geq 1.$$

Corollary 5.11. *If* $\mathcal{C}$ *is a Mengerian clutter. Then* $I(\mathcal{C})$ *has the strong persistence property.*

Proof. Its follows form Corollary 5.8 and Theorem 5.10 □

Example 5.12. *The edge ideal of* $\mathcal{C}_0$ *has the symbolic strong persistence property but fail the strong persistence property.*

6 Hibi ideals

Definition 6.1. *[7, Page 159] Let* $(P, \geq)$ *be a finite partially ordered set (a poset, for short) with* $P = \{p_1, \ldots, p_n\}$*. A poset ideal of* P *is a subset* $I \subseteq P$ *such that if* $p_i \in I$ *and* $p_j \leq p_i$*, then* $p_j \in I$*. To any poset ideal* I *of* P*, we associate the monomial*

$$u_I = \Big(\prod_{p_i \in I} x_i\Big)\Big(\prod_{p_i \notin I} y_i\Big) \in K[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

The set of all poset ideals of P *is denoted by* $\mathcal{J}(P)$*. We consider* $\emptyset \in \mathfrak{I}(P)$*. Then the* Hibi ideal *of* P *is the monomial ideal of* S *defined as*

$$H_P = (u_I \;:\; I \in \mathcal{J}(P)).$$

Theorem 6.2. *[4, Lemma 5.9] Let* H_P *be a Hibi ideal. Then*

$$\mathrm{Ass}(H_P^k) = \{(x_i, y_i) \;: p_i, p_j \in P, \quad p_i \leq p_j\}, \qquad \forall\, k \geq 1.$$

Definition 6.3. *Let* $I = (x^{e_1}, \ldots, x^{e_q}) \subseteq K[x_1, \ldots, x_n]$ *be a square-free monomial ideal with exponent vectors* $e_j \in \{0,1\}^n$*. The* Alexander dual *of* I *is the monomial ideal*

$$I^{\vee} := \bigcap_{j=1}^{q} (x_i \mid (e_j)_i = 1).$$

Equivalently, $I^{\vee}$ *is generated by monomials corresponding to minimal vertex covers of the clutter associated to* I*.*

Corollary 6.4. *[7, Lemma 9.1.9] Let* H_P *be a Hibi ideal. Then*

$$H_P^{\vee} = (x_i y_j \;:\; p_i, p_j \in P, \quad p_i \leq p_j), \qquad .$$

Example 6.5. *[23, Exercise 7.7.27] Let*

$$R = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$$

and consider the monomial ideal

$$I = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_6,\ x_6x_7,\ x_7x_8,\ x_8x_9,\ x_9x_{10},\ x_{10}x_{11},$$
$$x_{11}x_{12},\ x_5x_{12},\ x_3x_7,\ x_4x_6,\ x_4x_9,\ x_5x_{10},\ x_1x_9,\ x_2x_8,\ x_7x_{12},\ x_8x_{11}).$$

$J = I^{\vee}$ its Alexander dual. Then, as shown using `Macaulay2`*, the inclusion*

$$\mathrm{Ass}(R/J^3) \nsubseteq \mathrm{Ass}(R/J^4)$$

does not hold. *Hence, the Alexander dual J of I but fails the persistence property.*

This example is of particular importance because it disproves a conjecture that was once considered plausible: that a clutter has the strong persistence property if and only if its Alexander dual also satisfies it. If this conjecture were true, then by Corollary 6.4 we would immediately deduce that every Hibi ideal possesses the strong persistence property. However, the counterexample above shows that this dual equivalence fails in general, revealing that the strong persistence property is not preserved under Alexander duality. Consequently, the relationship between a clutter and its dual must be treated with greater care while their combinatorial structures are closely intertwined, persistence reflects deeper algebraic behavior that is not necessarily symmetric.

Theorem 6.6. *[4, Lemma 5.9] Every Hibi ideal H_P associated to a finite partially ordered set P satisfies:*

1. *$H_P^{(k)} = H_P^k$, for all $k \geq 1$. Furthermore, H_P is a* Mengerian *ideal.*
2. *$\mathrm{Ass}(R/H_P^k) = \mathrm{Ass}(R/H_P)$, for all $k \geq 1$.*

In particular, the set of associated primes of H_P^k stabilizes immediately at $k = 1$.

Theorem 6.7. *Every Hibi ideal H_P satisfies both the* persistence property *and the* strong persistence property*; that is,*

$$\mathrm{Ass}(R/H_P^t) \subseteq \mathrm{Ass}(R/H_P^{t+1}) \quad \textit{and} \quad H_P^{t+1} : H_P = H_P^t \quad \textit{for all } t \geq 1.$$

Proof. This result follows directly from Corollary 5.8, which establishes that any monomial ideal that is normally torsion-free necessarily satisfies the strong persistence property. From Theorem 6.6, they automatically fulfill the persistence property. □

This connection provides a clear algebraic bridge between the combinatorial structure of partially ordered sets and the persistence behavior of their corresponding edge ideals. In particular, it highlights that persistence in Hibi ideals is not accidental but a structural consequence of their normality and Mengerian nature, reinforcing the idea that the strong persistence property is an inherent and robust feature within this important class of monomial ideals.

Remark 6.8. Let n be a natural number and let $P_n = \{p_1, \dots, p_n\}$. Consider a partially ordered set $(P_{n+1}, \leq)$; without loss of generality, we may assume that p_{n+1} is a maximal element. Define

$$I_{n+1} = \{\, p_i \in P_{n+1} \mid p_i \leq p_{n+1} \,\}.$$

It is clear that I_{n+1} is the unique ideal of $(P_{n+1}, \leq)$ containing p_{n+1}. Then we have that

$$I \in \mathcal{J}(P_n) \quad \text{if and only if} \quad I \in \mathcal{J}(P_{n+1}) \setminus \{I_{n+1}\}.$$

Moreover, note that $I_{n+1} \setminus \{p_{n+1}\} \in \mathcal{J}(P_n)$. Hence,

$$G(H_{P_{n+1}}) = \{\, my_{n+1} \mid m \in G(H_{P_n}) \,\} \cup \{\, u_{I_{n+1}} \,\}.$$

Therefore, the set $\{my_{n+1} \mid m \in G(H_{P_n})\}$ forms a cone over $G(H_{P_n})$, and we obtain the relation

$$H_{P_{n+1}} = y_{n+1} H_{P_n} + (u_{I_{n+1}}).$$

Now, since both $H_{P_{n+1}}$ and H_{P_n} satisfy the strong persistence property, it is natural to ask whether this behavior extends to more general constructions. In particular, we may consider operations such as the *Hibi contraction* and the *Hibi lifting*, and investigate whether these operations preserve the strong persistence property when moving between levels that is, when descending or ascending in the poset hierarchy. This question suggests a deeper structural stability of the persistence phenomena within families of Hibi ideals.

7 Square-free monomial ideals in $K[x_1, x_2, x_3, x_4, x_5]$ and the strong persistence property

In this section, we discuss the fact that every square-free monomial ideal in the polynomial ring $K[x_1, x_2, x_3, x_4, x_5]$ over a field K satisfies the strong persistence property. To establish this result, we begin with the following definitions and notations, which will be used throughout the section.

Proposition 7.1. *[2, Proposition 3.5] Suppose that $\mathcal{C}$ is a strongly persistent clutter, $x \notin V(\mathcal{C})$, and $\mathcal{C}x$ is the cone over $\mathcal{C}$. Then $\mathcal{C}'$ is a strongly persistent clutter with $V(\mathcal{C}') = V(\mathcal{C}) \cup \{x\}$ and $E(\mathcal{C}') = \{V(\mathcal{C})\} \cup E(\mathcal{C}x)$.*

This result is particularly useful for our purpose of studying the families of clutters that satisfy the strong persistence property across different vertex levels n. It provides a constructive mechanism to move between levels while preserving the property under conic extensions, allowing a systematic comparison of the distribution of strongly persistent clutters as the number of vertices increases. Moreover, this proposition can be regarded as a generalization of Theorem 4.6, since it extends the structural preservation of the strong persistence property from specific configurations to a broader class of clutter transformations.

Proposition 7.2. *[2, Proposition 3.6] Let X be a finite set and let $x, y \notin X$ with $x \neq y$. Suppose that $\mathcal{C}$ is a clutter on $X \cup \{x, y\}$ such that for every $e \in E(\mathcal{C})$, either $x \in e$ or $y \in e$, but $\{x, y\} \nsubseteq e$. Then the following statements hold:*

(i) The clutter $\mathcal{C}'$ is strongly persistent, where $E(\mathcal{C}') = E(\mathcal{C}) \cup \{\{x, y\}\}$.

(ii) The clutter $\mathcal{C}'$ is strongly persistent, where $E(\mathcal{C}') = E(\mathcal{C}) \cup \{X, \{x, y\}\}$.

This result is of particular importance for our study, since it provides a way to move between different levels of clutters. On one hand, it allows us to analyze families of clutters by ascending from level $n-1$ to level $n+1$, and, when combined with the elimination and contraction results, to descend again to level n. In this sense, one can reach level n through two distinct paths: by lifting from $n-1$ or by projecting from $n+1$. This dual approach offers refined control and sharper bounds for the distribution of clutters satisfying the strong persistence property. On the other hand, this result also contributes to the study of suspension operations and provides insight into the conjecture concerning them stated in Conjecture 4.8.

Proposition 7.3. *[2, Proposition 3.7] Let $\mathcal{C}$ be a strongly persistent clutter on a finite set X, and let $x, y \notin X$ with $x \neq y$. Then the clutter $\mathcal{C}'$ is defined by*

$$V(\mathcal{C}') = X \cup \{x, y\} \quad \text{and} \quad E(\mathcal{C}') = \{\, e \cup \{x\},\, e \cup \{y\} \mid e \in E(\mathcal{C}) \,\}$$

is strongly persistent.

The following lemmas and propositions address particular cases for clutters with $n = 5$ vertices. They play a key role in establishing the result asserting that every clutter defined on five vertices satisfies the strong persistence property. In addition, these results contribute to our ongoing objective of controlling the level-lifting process within families of clutters. Specifically, they allow us to analyze and bound the cardinalities of such families when moving from a set of vertices $[n]$ to the higher levels $[n+3]$, $[n+4]$, or $[n+5]$. This perspective provides a clearer understanding of how the strong persistence property behaves as we increase the number of vertices, and it becomes an essential tool for estimating and comparing the growth of strongly persistent clutters across different dimensions.

The interaction between these results and the operations of vertex elimination and contraction is especially relevant. By combining these two descending operations with the ascending constructions described earlier, we can reach the same level n through two distinct paths either by lifting from level $n-1$ or by descending from level $n+1$. This dual approach not only reinforces the structural understanding of persistence within clutters but also sharpens the bounds for the families $\mathfrak{S}_n$ and $\mathfrak{C}_n$, providing a clearer picture of how strong persistence behaves across dimensions.

lemma 7.4. *[2, Lemma 3.8] Let $X = \{x, y, x_1, x_2, x_3\}$ be a finite set of cardinality 5, and let $\mathcal{C}$ be a clutter on X satisfying the following conditions:*

(i) Every edge $e \in E(\mathcal{C})$ has cardinality $|e| = 3$;

(ii) $\{x, y\} \nsubseteq e$ for all $e \in E(\mathcal{C})$;

(iii) $\{x_1, x_2, x_3\} \notin E(\mathcal{C})$.

Then $\mathcal{C}$ is strongly persistent.

lemma 7.5. *[2, Lemma 3.9] Let $X = \{x, y, x_1, x_2, x_3\}$ be a finite set of cardinality 5, and let $\mathcal{C}$ be a clutter on X such that for every $e \in E(\mathcal{C})$ the following conditions hold:*

(i) $|e| = 3$;

(ii) $\{x, y\} \nsubseteq e$;

(iii) $\{x_1, x_2, x_3\} \in E(\mathcal{C})$.

Then $\mathcal{C}$ is strongly persistent.

lemma 7.6. *[2, Lemma 3.10] Let $\mathcal{C} = (V, E)$ be a clutter with $V(\mathcal{C}) = \{x, y, x_1, x_2, x_3\}$ satisfying the following conditions:*

(1) $|e| = 3$ *for each* $e \in E(\mathcal{C})$;

(2) There exists a unique edge $e \in E(\mathcal{C})$ *such that* $\{x, y\} \subseteq e$;

(3) $e \cap \{x, y\} \neq \emptyset$ *for each* $e \in E(\mathcal{C})$.

Then the following statements hold.

(i) $\mathcal{C}$ *is strongly persistent.*

(ii) $\mathcal{C}'$ *is strongly persistent, where* $E(\mathcal{C}') = E(\mathcal{C}) \cup \{\{x_1, x_2, x_3\}\}$.

Proposition 7.7. *[2, Proposition 3.11] Let $\mathcal{C}$ be a clutter with $E(\mathcal{C}) = \{e_1, e_2, e_3, \ldots, e_r\}$ such that $e_1 = \{x, y\}$ is an edge of cardinality two, and there exists an edge e_2 satisfying $e_1 \cap e_2 = \emptyset$, while $e_j \cap e_1 \neq \emptyset$ for each $3 \leq j \leq r$. Then $\mathcal{C}$ is strongly persistent.*

The significance of Proposition 7.7 lies in the fact that it allows us to study clutters that belong to levels two units above a given configuration. By identifying an edge of size two that interacts with all remaining edges except for one disjoint edge, this result provides a constructive criterion to lift clutters from level n to level $n+2$ while preserving the strong persistence property. This lifting process is particularly relevant because it offers a way to connect the structural behavior of small clutters with that of higher-dimensional families, thereby facilitating the recursive study of how the strong persistence property behaves under successive enlargements of the vertex set.

Moreover, Proposition 7.7 can be viewed as a natural generalization of Proposition 7.2, since it extends the configuration of two distinguished vertices x and y to a broader setting where the connectivity pattern among the remaining edges is controlled combinatorially. In this sense, Proposition 7.7 are not only strengthens the scope of Proposition 7.2 but also provides a structural bridge between local two-vertex interactions and global persistence phenomena in higher levels of the clutter hierarchy.

Corollary 7.8. *[2, Proposition 3.12] Let $\mathcal{C}$ be a clutter on $X = \{x_1, x_2, x_3, x_4, x_5\}$ such that $2 \leq |e| \leq 3$ for each $e \in E(\mathcal{C})$, and $\mathcal{C}$ has only two edges e_1, e_2 with $|e_1| = |e_2| = 2$ and $e_1 \cap e_2 = \emptyset$. Then $\mathcal{C}$ is strongly persistent.*

This result is particularly useful as it provides an explicit construction of a strongly persistent clutter from two disjoint vertex sets, allowing the inclusion of both controlled bipartite relations (through the subset $A \subseteq X$) and a complete edge X that ensures the propagation of persistence. In particular, Corollary 7.8 extends Proposition 4.3 by generalizing the notion of a simple cone to a richer structure that combines bipartite and complete components while preserving strong persistence in a broader setting. This generalization is valuable for generating new families of clutters with stability properties and for studying the interaction between bipartite and complete substructures within the same algebraic framework.

Definition 7.9. *[20, Definition 1.1] A* matroid *$\mathcal{M}$ is an ordered pair $\mathcal{M} = (E, \mathcal{I})$, where E is a finite set, called the* ground set, *and $\mathcal{I}$ is a nonempty collection of subsets of E, called the* independent sets, *satisfying the following axioms:*

(I1) $\emptyset \in \mathcal{I}$;

(I2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$ (hereditary property);

(I3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element $x \in J \setminus I$ such that $I \cup \{x\} \in \mathcal{I}$ (exchange property).

The maximal elements of $\mathcal{I}$ under inclusion are called the bases *of the matroid. All bases of a matroid have the same cardinality, which is called the* rank *of the matroid.*

lemma 7.10. *[2, Lemma 3.16] Let $\mathcal{C}$ be a clutter on the vertex set $\{x_1, x_2, x_3, x_4, x_5\}$ such that $|e| = 3$ for every $e \in E(\mathcal{C})$. If any two distinct vertices $x_i, x_j \in V(\mathcal{C})$ are contained together in at least two distinct edges, then $E(\mathcal{C})$ forms the collection of bases of a matroid. Consequently, $\mathcal{C}$ is strongly persistent.*

Proposition 7.11. *[2, Proposition 3.17] Let $\mathcal{C}$ be a clutter on $\{x_1, x_2, x_3, x_4, x_5\}$ such that $|e| = 3$ for each $e \in E(\mathcal{C})$. Then $\mathcal{C}$ is strongly persistent.*

Theorem 7.12. *[2, Theorem 3.18] Every square-free monomial ideal in $K[x_1, x_2, x_3, x_4, x_5]$ has the strong persistence property.*

Theorem 7.12 is fundamental for the study of clutters, since it establishes that every square-free monomial ideal in five variables satisfies the strong persistence property. This result provides a powerful reduction principle: to understand the behavior of persistence in general, it suffices to analyze the case $n > 5$. Consequently, it allows us to identify families of clutters with the SPP more efficiently and to refine bounds on the number of persistent clutters by relating them to the behavior of the five variable case. In particular, this theorem highlights that the smallest clutter failing the property is precisely $\mathcal{C}_0$, which plays a key role in the structure theory of non-persistent configurations.

8 Low and Upper bound for $|\mathfrak{S}_n|$

In this section, we focus on establishing bounds for the cardinality of $\mathfrak{S}_n$, the family of clutters satisfying the strong persistence property. Our approach is based on the results developed in the survey, where several algebraic and combinatorial constructions were introduced to compare families of clutters at consecutive levels $n-1$, n, and $n+1$. These comparisons allow us to estimate the growth and distribution of strongly persistent clutters through recursive inequalities and level transitions.

Proposition 8.1. *$(n+1)\,|\mathfrak{C}_n| \; + |\mathfrak{S}_n| \;\leq\; |\mathfrak{S}_{n+1}|$, for $n > 2$*

Proof. Given $\mathcal{C} = ([n], E) \in \mathfrak{C}_n$, we consider the clutters

$$\mathcal{C}' := ([n+1],\, E(\mathcal{C}) \cup \{\{n+1\}\}) \quad \text{and} \quad \mathcal{C}'_i \text{ for } i \in V(\mathcal{C}),$$

where each $\mathcal{C}'_i$ is defined as in Corollary 4.13, for every $i \in [n]$. From Proposition 3.4 and Corollary 4.13, these clutters have the persistence property.

Finally, if $\mathcal{D} = ([n], F) \in \mathfrak{S}_n$, then we set

$$\bar{\mathcal{D}} := ([n+1], F) \in \mathfrak{S}_{n+1}.$$

Since $\mathcal{C}', \mathcal{C}'_i$ and $\bar{\mathcal{D}}$ are pairwise distinct clutters in $\mathfrak{S}_{n+1}$, and there are $(n+1)\,|\mathfrak{C}_n| + |\mathfrak{S}_n|$ of them, we obtain

$$(n+1)\,|\mathfrak{C}_n| + |\mathfrak{S}_n| \;\leq\; |\mathfrak{S}_{n+1}|.$$

□

Remark 8.2. *In the above proposition, if Conjecture 4.8 holds, we additionally obtain that $\mathcal{C}_x$ satisfies the strong persistence property. Hence,*

$$(n+3)\,|\mathfrak{C}_n| \; + \; |\mathfrak{S}_n| \;\leq\; |\mathfrak{S}_{n+1}|.$$

The results presented in the survey are particularly significant because the more operations we can define to move upward or downward between levels, the sharper our bounds on $|\mathfrak{S}_n|$ become. Each new operation preserving (or characterizing the loss of) the strong persistence property contributes to a finer understanding of the structure and asymptotic behavior of the families of clutters across dimensions

Proposition 8.3. *$|\mathfrak{C}_{n+1}| \leq |\mathfrak{C}_n|^2$, for $n > 0$.*

Proof. Let $\mathcal{C} = ([n+1], E) \in \mathfrak{C}_{n+1}$ and we define

$$E' = \{\, e \in E \mid n+1 \in e \,\}, \qquad E'' = \{\, e \in E \mid n+1 \notin e \,\},$$

and

$$E''' = \{\, e \setminus \{n+1\} \mid e \in E' \,\},$$

Is clear that following its holds

(i) E' is a clutter on $[n+1]$ in which every edge contains $n+1$.

(ii) The family E''' is a clutter on $[n]$, and E' is precisely the cone over E''' ($E' = E'''n+1$).

(iii) E'' is the edge set of the deletion $\mathcal{C} \setminus \{n+1\}$, hence $E'', E''' \subseteq \mathfrak{C}_n$.

Consequently, since $E = E' \cup E''$ and $|E'''| = |E'|$, we have

$$\max\{\,|E'|, |E''|, |E'''|\,\} \leq |\mathfrak{C}_n|, \qquad \text{and therefore} \qquad |\mathfrak{C}_{n+1}| \leq |\mathfrak{C}_n|^2.$$

□

This decomposition expresses every clutter on $n+1$ vertices as a combination of a deletion and a cone derived from clutters on n vertices. Hence, the number of possible clutters in level $n+1$ is bounded by the square of those in level n. This bound is rough but useful, as it gives a recursive framework that helps control the growth of $\mathfrak{C}_n$ across levels.

Proposition 8.4. $2\,|\mathfrak{N}_n| \;\leq\; |\mathfrak{N}_{n+1}|$ *for* $n > 0$

Proof. By Theorem 4.6, the cone operation preserves the failure of the strong persistence property; that is, for every $\mathcal{C} \in \mathfrak{N}_n$ and every new vertex $z \notin V(\mathcal{C})$, one has

$$\mathcal{C} \in \mathfrak{N}_n \iff \mathcal{C}z \in \mathfrak{N}_{n+1}.$$

Define the two injective maps

$$\Phi_x(\mathcal{C}) = \mathcal{C}x, \qquad \Phi_y(\mathcal{C}) = \mathcal{C}y.$$

Since the vertices x and y are distinct and external to $V(\mathcal{C})$, the resulting clutters $\Phi_x(\mathcal{C})$ and $\mathcal{C}$ belong to disjoint copies of $\mathfrak{N}_n$ inside $\mathfrak{N}_{n+1}$. Each Φ_z is injective because contracting the vertex z recovers $\mathcal{C}$. Thus, there exist two disjoint embeddings of $\mathfrak{N}_n$ into $\mathfrak{N}_{n+1}$, which gives

$$2\,|\mathfrak{N}_n| \leq |\mathfrak{N}_{n+1}|.$$

□

Proposition 8.5. *We can decompose the family* $\mathfrak{N}_n$ *as a disjoint union*

$$\mathfrak{N}_n = \bigcup_{r=1}^{\binom{n}{\lfloor n/2 \rfloor}} \mathfrak{N}_{n,r},$$

where $\mathfrak{N}_{n,r}$ *denotes the subset of* $\mathfrak{N}_n$ *consisting of all clutters* $\mathcal{C}$ *satisfying* $|E(\mathcal{C})| = r$ *and*

$$\sum_{r=1}^{\binom{n}{\lfloor n/2 \rfloor}} |\mathfrak{N}_{n,r}|\, 2^r \;\leq\; |\mathfrak{N}_{n+1}|.$$

Proof. Given $\mathcal{C} = ([n], E) \in \mathfrak{N}_n$ and a subset $E_1 \subseteq E$, let $E_2 = E \setminus E_1$, and define the clutter

$$\mathcal{C}_{E_1}^{E_2} = ([n+1],\, E_1 \cup E_2x),$$

where $x = n+1$ and $E_2x = \{e \cup \{x\} \mid e \in E_2\}$ denotes the cone over E_2.

By Theorem 4.6, if $\mathcal{C} \in \mathfrak{N}_n$ then $\mathcal{C}_{E_1}^{E_2} \in \mathfrak{N}_{n+1}$. Indeed, if $\mathcal{C}_{E_1}^{E_2}$ were strongly persistent, then contracting the vertex x would yield $\mathcal{C} = (\mathcal{C}_{E_1}^{E_2})_{/x}$, implying $\mathcal{C} \in \mathfrak{S}_n$, a contradiction. Therefore, $\mathcal{C}_{E_1}^{E_2} \in \mathfrak{N}_{n+1}$.

Moreover, if $E(\mathcal{C}_{E_1}^{E_2}) = E(\mathcal{C}_{E_1'}^{E_2'})$, then necessarily $E_2x = E_2'x$, hence $E_2 = E_2'$ and $E_1 = E_1'$. Thus, distinct decompositions of E yield distinct clutters.

Consequently, for each clutter $\mathcal{C} \in \mathfrak{N}_{n,r}$, there exist 2^r distinct clutters of the form $\mathcal{C}_{E_1}^{E_2}$, indexed by all subsets $E_1 \subseteq E$. These 2^r clutters form pairwise disjoint copies of $\mathfrak{N}_{n,r}$ embedded in $\mathfrak{N}_{n+1}$. Hence,

$$\sum_{r=1}^{\binom{n}{\lfloor n/2 \rfloor}} |\mathfrak{N}_{n,r}|\, 2^r \;\leq\; |\mathfrak{N}_{n+1}|.$$

□

9 Discussion and questions

Throughout this survey, we have explored the structural and algebraic mechanisms that govern the strong persistence property (SPP) in clutters. A central goal has been to understand and estimate the quantities

$$|\mathfrak{S}_n|, \quad |\mathfrak{N}_n|, \quad |\mathfrak{C}_n|,$$

representing respectively the number of clutters on $[n]$ satisfying SPP, those failing SPP, and the total number of clutters. The results presented so far give us both *ascending* constructions (such as cones or vertex extensions) and *descending* ones (like vertex deletions or contractions) that preserve or recover persistence. These tools together provide a framework to establish lower and upper bounds for $\mathfrak{S}_n$ in terms of quantities depending on levels $n-1$ and $n+1$. we anticipate that operations such as the *corona product*, various *vertex-duplication schemes*, and other combinatorial extensions could be used to improve these bounds and generate recursive estimates. On the other hand, the study of deletions and contractions suggests a possible approach to classify the minimal obstructions to persistence.

Is there a characterization of clutters failing the SPP in terms of forbidden embeddings? In other words, does there exist a finite list of minimal clutters $\{\mathcal{C}_1, \ldots, \mathcal{C}_t\}$ containing, for instance, the Villarreal clutter $\mathcal{C}_0$ such that a clutter $\mathcal{C}$ fails the SPP if and only if it contains one of the $\mathcal{C}_i$ as an induced subclutter?

Can analogous embeddings or structural characterizations be established for the weaker persistence property (PP)? Understanding whether similar minimal obstructions exist for PP would not only clarify the hierarchy between these two properties but also provide a unified combinatorial perspective on persistence phenomena in monomial ideals.

In order to refine the current upper and lower bounds for $|\mathfrak{S}_n|$, it becomes crucial to understand more precisely the algebraic and combinatorial operations that control the transition between consecutive levels. Some operations preserve the strong persistence property when moving upward (such as cones or corona products), while others maintain it when moving downward (like deletions and contractions). However, the extent to which these and other operations can fully describe the propagation of persistence across dimensions remains unclear.

Which additional operations should be studied to achieve a finer control over the ascending and descending behavior of persistence, and consequently improve the known bounds for $|\mathfrak{S}_n|$? Specifically, can we identify new constructions such as generalized cone operations, layered corona products, or restricted vertex duplications that not only preserve the SPP/PP but also yield predictable changes in the number of persistent clutters across levels? Developing a systematic catalogue of such operations could lead to recursive formulas or asymptotic estimates for $|\mathfrak{S}_n|$, $|\mathfrak{N}_n|$, and $|\mathfrak{C}_n|$, providing a deeper quantitative understanding of persistence in the combinatorial landscape of clutters.

These questions point toward a broader research direction aiming to describe persistence properties via structural patterns and embedding relations, connecting combinatorial configurations with the algebraic behavior of their corresponding edge ideals.

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