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Essential ideal transforms

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ABSTRACT

It is our intention in this research to generalize some concepts in local cohomology such as free modules, contravariant functor *ext*, covariant functor *Ext* and ideal transforms with e-exact sequences. The e-exact sequence was introduced by Akray and Zebari [1] in 2020. We prove that essential free module is an essential projective and a submodule rM of M is a quotient of an essential free module. Furthermore, we obtain that for a torsion-free modules B, ${}_e ex^n_R(P, B) = 0$ whereas ${}_e Ext^n_R(A, E) = 0$ for every R-module A. Also for any torsion-free modules we have an e-exact sequence $0 \to \Gamma_a(B) \to B \to D_a(B) \to H^1_a(B) \to 0$ and an isomorphism between B and $rD_a(B)$. Finally we generalize Mayer-Vietoris with e-exact sequences in essential local cohomology, we obtain a special e-exact sequence.

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1 Introduction

Throughout this article, R will denote a Noetherian domain and B torsion free e-enjective R-module. The Mayer-Vietoris sequence involves two ideals so throughout this research b will denote a second ideal. In 1972, R. S. Mishra introduced a generalization for split

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sequence where a semi-sequence $M_{i-1} \overset{f_{i-1}}{\to} M_i \overset{f_i}{\to} M_{i+1}$ is called semi-split if $Ker(f_i)$ is a direct summand of M_i [8]. So a semi-split is split if and only if it is exact. In 1999, Davvaz and parnian-Goramaleky introduced a generalization for exact sequences called it a U-exact sequence [6]. A submodule N of an R-module M is called essential or large in M if it has non-zero intersection with every non-zero submodule of M and denoted by $N \leqslant_e M$. Akray and Zebari in 2020 [1] introduced another generalization to exact sequences of modules and instead of the equality of $Im(f)$ with $Ker(g)$ they took $Im(f)$ as a large (essential) submodule of $Ker(g)$ in a sequence $0 \to A \overset{f}{\to} D \overset{g}{\to} C \to 0$ and called it essential exact sequence or simply e-exact sequence. Equivalently, a sequence of R-modules and R-morphisms $\cdots \to N_{i-1} \overset{f_{i-1}}{\to} N_i \overset{f_i}{\to} N_{i+1} \to \cdots$ is said to be essential exact (e-exact) at N_i, if $Im(f_{i-1}) \leqslant_e Ker(f_i)$ and to be e-exact if it is e-exact at N_i for all i. In particular, a sequence of R-modules and R-morphisms $0 \to L \overset{f_1}{\to} M \overset{f_2}{\to} N \to 0$ is a short e-exact sequence if and only if $Ker(f_1) = 0, Im(f_1) \leqslant_e Ker(f_2)$ and $Im(f_2) \leqslant_e N$. The authors of [1] studied some basic properties of e-exact sequences and established their connection with notions in module theory and homological algebra [2]. Also, F. Campanini and A. Facchini have worked on e-exact sequences and studied the relation of e-exactness with some related functors like the functor defined on the category of R-modules to the spectral category of R-modules and the localization functor with respect to the singular torsion theory [5]. Furthermore, Akray and R. Mustafa in 2023 introduced and proved further properties of e-exact sequences and we will restrict our discussion to their applications on both injective modules and the torsion functor of local cohomology [3]. "Local cohomology was introduced by Grothendieck in a 1961 Harvard seminar and later published by Hartshorne in 1967. Next, this subject was studied by Hartshorne and numerous authors even in the recent years see [9], [4] and [7].

In this research we generalize some concepts in local cohomology such as contravariant functor ${}_eext$, covariant functor ${}_eExt$ and ideal transforms with e-exact sequences.

In section two, we defined essential free modules (briefly e-free) and we present the basic theory of essential free modules such as let P be a torsion-free module of A_1. If a submodule rP is an e-projective then every e-exact sequence e-split. A submodule rM of M is a quotient of an e-free R-module. Also we describe the concept ${}_eext_n^R$ and ${}_eExt_n^R$ as well as we characterize the properties of each of them. For example, let $0 \to A' \to A \to A'' \to 0$ be an e-exact sequence of R-modules , then there is a long e-exact sequence $0 \to Hom(A'', B) \to Hom(A, B) \to Hom(A', B) \to {}_eext_R^1(A'', B) \to {}_eext_R^1(A, B) \to \cdots$ and also we have for any R-module A and any e–injective R-module E, ${}_eExt_R^n(A, E) = 0$, for all $n \geq 1$.

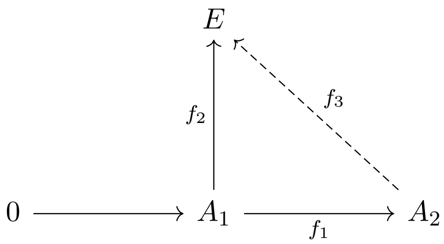
In section three, we construct essential ideal transforms and we find the new e-exact sequence by generalized the idea of Mayer-vietores sequence. Also we prove that for any torsion-free R-module B there exists $0 \neq r \in R$ such that $\epsilon^* : B \to rD_a(B)$ is an isomorphism if and only if $\Gamma_a(B) = H_a^1(B) = 0$ and also we show that there is an e-exact sequences $0 \to D_{r(a+b)}(B) \to D_a(B) \bigoplus D_b(B) \to D_{a\cap b}(B) \to rH_a^2 D_{r(a+b)}(B) \to rH_a^2 D_a(B) \bigoplus rH_a^2 D_b(B) \to rH_a^2 D_{a\cap b}(B) \to \cdots$.

2 contravariant and covariant right essential derived functor

2.1 Contravariant essential derived functor

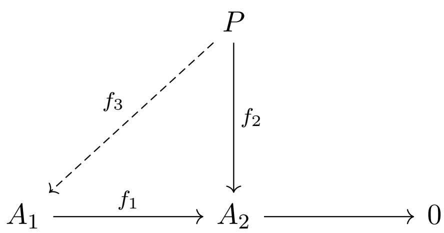
In this subsection we want to describe contravariant right derived functor ext^n_R on the e-projective resolution call it essential derived functor (beriefly ${}_e ext^R_n$) and discuss some properties of them. On the other hand We present some definition that are central for our object such as essential injective, essential projective and essential free modules as the following:

Definition 2.1. *An R-module E is an e-injective if it satisfies the following condition: for any monic $f_1 : A_1 \to A_2$ and any map $f_2 : A_1 \to E$, there exist $0 \neq r \in R$ and $f_3 : A_2 \to E$ such that $f_3 \circ f_1 = rf_2$.*



In this case, we say the map f_3 is essentially extends to the map f_2.

Definition 2.2. *An R-module P is e-projective if it satisfies the following condition: for any e-epic map $f_1 : A_1 \to A_2$ and any map $f_2 : P \to A_2$, there exist $0 \neq r \in R$ and $f_3 : P \to A_1$ such that $f_1 \circ f_3 = rf_2$.*



The following example shows that an e-projective module may not be projective.

Example 2.3. *Consider the e-exact sequence*

$$0 \to 4\mathbb{Z} \overset{f_1}{\to} \mathbb{Z} \overset{f_2}{\to} \frac{\mathbb{Z}}{4\mathbb{Z}} \to 0,$$

where $f_1(x) = 2x$ and $f_2(x) = 2x + 4\mathbb{Z}$ is an e-split, because we have a map $f : \mathbb{Z} \to 4\mathbb{Z}, f(x) = 4x$ such that $f \circ f_1(x) = f(f_1(x)) = f(2x) = 8x = 8I_{\mathbb{Z}}$. Thus we get $\mathbb{Z}/4\mathbb{Z}$ is an e-projective by[1, Proposition 3.2], while $\mathbb{Z}/4\mathbb{Z}$ is not projective.

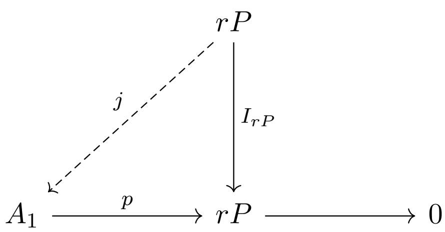
Definition 2.4. *A submodule S of M is an e-retract of M if there exist a map $P : M \to S; P(s) = s$ for all $s \in S$ called an e-retraction and $0 \neq r \in R$ such that $Pi = rI_S$, where $i : S \to M$ is the inclusion.*

Corollary 2.5. *Let S be a torsion-free submodule of M. A submodule rS is a direct summand of rM if there exist $0 \neq r \in R$ and a retraction $p : M \to S$*

Proof. Let $i : S \to M$ be the inclusion we show that $rM = rS \oplus T$, where $T = rKerP$. If $rm \in rM$ then $rm = (rm - rpm) + rpm$. Obviously, $r(p(m - pm)) = p(rm - rpm) = rpm - rppm = 0$, because $pm \in S$ and so $ppm = pm$, on the other hand $rpm \in rImp = rS$. Therefore, $rM = rS + T$. If $rm \in rS$, then $rpm = rm$; if $rm \in T = rKerp$, then $pm = 0$. Hence, if $rm \in rS \cap T$, then $m = 0$. Therefore, $rS \cap T = 0$ and $rM = rS \oplus T$. □

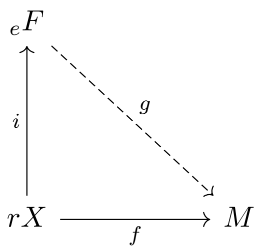
Proposition 2.6. *Let P be a torsion-free submodule of A_1. If a submodule rP is e-projective, then every e-exact sequence ending with rP e-splits.*

Proof. If rP is an e-projective then there exist $j : rP \to A_1$ and $0 \neq r \in R$ such that $pj = rI_{rP}$. As the following diagram:



By Corollary 2.5 the submodule rP is a direct summand of rA_1, then the proof is complete by using [3, Proposition 2.5] □

Definition 2.7. *Let M and ${}_eF$ be any R-module, $0 \neq r \in R$ and X be a basis of ${}_eF$. We call ${}_eF$ is an e-free if for any map $f : rX \to M$ and inclusion map $i : rX \to {}_eF$ there exists a unique map $g : {}_eF \to M$ such that $g \circ i = rf$. In this case, we say g essentially extend to f.*



Proposition 2.8. *Let M be any R-module and X be a basis of a free module ${}_eF$. Then ${}_eF$ is an e-free-module.*

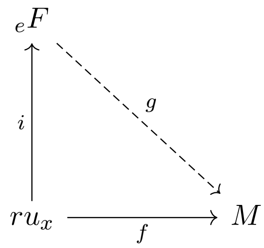
Proof. By [9, Proposition 2.34], for any map $f : X \to M$ and inclusion map $i : X \to {}_eF$ there exists a unique map $g : {}_eF \to M$ such that $g \circ i = rf$ since ${}_eF$ is a free module. Therefore, there is an inclusion map $h : rX \to_e F$ which is a composite of $i \circ p$ where

$p : rX \to X$ be an inclusion map. Thus $g \circ h(rx) = g(h(rx)) = g(i \circ p(rx)) = gi(p(rx)) = gi(rx) = rgi(x) = rf(x) = f(rx) = f(p(rx)) = f \circ p(rx)$. □

In the above Proposition we showed that every free module is an e-free module, but the converse is not true in general because, rX may not be a basis.

Theorem 2.9. *A submodule rM of M is a quotient of an e-free R-module.*

Proof. Let M be generated by the set X and ${}_eF$ be an e-free module with basis $u_x : x \in X$, then by definition there exist $0 \neq r \in R$ and a map $g : {}_eF \to M$ with $g(ru_x) = rx$ for all $x \in X$, so $Img = rM$ is a submodule of M containing rX, implies $rM \cong \frac{{}_eF}{Kerg}$ showing as the following diagram:



□

Theorem 2.10. *Let ${}_eF$ be an e-free R-module. If $p : A \to A''$ is surjective, then for every $h : {}_eF \to A''$, there exist $0 \neq r \in R$ and $g : {}_eF \to A$ such that $p \circ g = rh$.*

Proof. Let X be a basis of ${}_eF$. For each $x \in X$, $h(rx) \in A''$ has the form $h(rx) = p(ra_x)$ for some $a_x \in A$ because p is surjective by the axiom of choice there is a function $u : rX \to A$ with $u(rx) = ra_x$ for all $x \in X$ so by definition there exist a map $g : {}_eF \to A$ with $g(rx) = ra_x$ for all $x \in X$. Now, $p \circ g(rx) = p(g(rx)) = p(ra_x) = h(rx)$. □

Theorem 2.11. *Let P torsion-free R-module. If a submodule rP is an e-projective then rP is a direct summand of a submodule of an e-free R-module for $0 \neq r \in R$.*

Proof. Assume that rP is an e-projective. By Theorem 2.9 rP is a quotient of an e-free module ${}_eF$. Thus there is an e-exact sequence $0 \to kerg \to {}_eF \overset{g}{\to} rP \to 0$ which is an e-split by Proposition 2.6. Therefore $r_eF \cong kerg \oplus rP$ by [3, Proposition 2.5]. □

Definition 2.12. *An e-projective resolution of an $R-$module A is an e-exact sequence $\cdots \to P_{n+1} \to P_n \cdots \to P_1 \to P_0 \to A \to 0$ where each P_n is an e-projective R-module.*

Definition 2.13. *If T is a contravariant functor, then*

$$(R^nT)A = H^n(TP_A) = \frac{KerTd_{n+1}}{ImTd_n},$$

where $\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$ is the e-projective resolution of an R-module A. In particular, we put $T = Hom(\ ,B)$, define ${}_eext^n_R(\ ,B) = R^nT$. Then ${}_eext^n_R(A,B) = H^n(Hom_R(P_A,B))$, which means ${}_eext^n_R = \frac{Kerd^{n^*}}{Imd^{(n-1)*}}$, where $d^{n^*} : Hom(P_{n-1},B) \to Hom(P_n,B)$ is defined as usual by $d^{n^*} : f \longmapsto fd^n$.

Theorem 2.14. *Let A be any R-modules. Then ${}_e ext^n_R(A,B) = 0$ for all negative integer n.*

Proof. Suppose that $\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0 : P$ be an e-projective resolution for A. Then the deleted complex of A is $\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0 : P_A$, after applying $Hom(\ ,B)$ on the deleted complex, we get $0 \to Hom(P_0,B) \to Hom(P_1,B) \to Hom(P_2,B) \to \cdots$ by [1, Theorem 2.7], which implies that $Hom(P_n,B) = 0$ for all negative integer number n. Hence ${}_e ext^n_R = 0$ for all negative integer number n. □

Theorem 2.15. *Let A be any R-modules and $n = 0$. Then ${}_e ext^n_R(A,B) \cong Hom(A,B)$*

Proof. Let $\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ be an e-projective resolution for A. By definition

$$ {}_e ext^0_R(A,B) = H^0(Hom(P_A,B)) = \frac{Kerd_1^*}{Imd_\epsilon^*} = Kerd_1^*. $$

But left e-exactness of $Hom(\ ,B)$ gives an e-exact sequence

$$ 0 \to Hom(A,B) \xrightarrow{\epsilon^*} Hom(P_0,B) \xrightarrow{d_1^*} Hom(P_1,B) \xrightarrow{d_2^*} Hom(P_2,B) \to \cdots $$

by [3, Proposition 2.7]. We define $\epsilon^* : Hom(A,B) \to Kerd_1^*$, since $Im\epsilon^* \leqslant_e Kerd_1^*$, ϵ^* is well-defined and since $Hom(\ ,B)$ is left e-exact functor then ϵ^* is monic. Now, we want to prove that ϵ^* is an epic. Let $f \in Kerd_1^*$ where $f : P_0 \to B$ and $g \in Hom(A,B)$, then $d_1^*(f(p_0)) = f(d_1(p_1))$ by e-exactness there exist $0 \neq r \in R$ and $p_0 \in ker\epsilon$ such that $d_1(p_1) = rp_0$ so $rf(p_0) = rg\epsilon(p_0) = r\epsilon^*(g(a))$ implies $f(p_0) = \epsilon^*(g(a))$. Hence ϵ^* is an isomorphism because, ${}_e ext^n_R(A,B) = Kerd_1^*$ and ${}_e ext^n_R(A,B)$ is isomorphic to $Hom(A,B)$. □

Theorem 2.16. *Let P be an e-projective, then ${}_e ext^n_R(P,B) = 0$, for all $n \geq 1$*

Proof. Since p is an e-projective, the e-projective resolution is $0 \to P \xrightarrow{1_P} P \to 0$ which is 1_p. The corresponding deleted e-projective resolution P_P is $0 \to P \to 0$. By applying $Hom(\ ,B)$ to the deleted complex we obtain ${}_e ext^n_R(P,B) = 0$ for all $n \geq 1$. □

Corollary 2.17. *Let $0 \to A' \to A \to A'' \to 0$ be an e-exact sequence of R-modules , then there is a long e-exact sequence $0 \to Hom(A'',B) \to Hom(A,B) \to Hom(A',B) \to {}_e ext^1_R(A'',B) \to {}_e ext^1_R(A,B) \to \cdots$.*

Proof. By [2, Theorem 3.7], we have an e-exact sequence of deleted complexes $0 \to P''_{A'} \to P_A \to P'_{A''} \to 0$. If $T = Hom(\ ,B)$, then $0 \to TP'_{A''} \to TP_A \to TP''_{A'} \to 0$ is still e-exact by [3, Proposition 2.7]. Then by [2, Theorem 3.2] we have an e-exact sequence $0 \to H^0(Hom(P'_{A''},B)) \to H^0(Hom(P_A,B)) \to H^0(Hom(P''_{A'},B)) \to H^1(Hom(P'_{A''},B)) \to H^1(Hom(P_A,B)) \to \cdots$. By using the definition of ${}_e ext^R_n$, Theorem 2.15 and Theorem 2.14 we obtain an e-exact sequence $0 \to Hom(A'',B) \to Hom(A,B) \to Hom(A',B) \to {}_e ext^1_R(A'',B) \to {}_e ext^1_R(A,B) \to \cdots$. □

Theorem 2.18. *Given a commutative diagram of R-modules having e-exact rows as the following:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{p} & A'' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & C' & \xrightarrow{j} & C & \xrightarrow{q} & C'' & \longrightarrow & 0, \end{array}$$

then there is a commutative diagram of R-modules with e-exact rows:

$$\begin{array}{ccccccc} {}_eext^n_R(A'',B) & \xrightarrow{p*} & {}_eext^n_R(A,B) & \xrightarrow{i^*} & {}_eext^n_R(A',B) & \xrightarrow{\sigma^n} & {}_eext^{n+1}_R(A'',B) \\ h^*\uparrow & & g^*\uparrow & & f^*\uparrow & & h^*\uparrow \\ {}_eext^n_R(C'',B) & \xrightarrow{q*} & {}_eext^n_R(C,B) & \xrightarrow{j^*} & {}_eext^n_R(C',B) & \xrightarrow{\sigma'^n} & {}_eext^{n+1}_R(C'',B) \end{array}$$

Proof. By [2, Theorem 3.7], we have an e-exact sequence of deleted complexes $0 \to P''_{A'} \to P_A \to P'_{A''} \to 0$. If $T = Hom(\ ,B)$, then $0 \to TP'_{A''} \to TP_A \to TP''_{A'} \to 0$ is still e-exact by [3, Proposition 2.7]. By [2, Remark 3.3] there is a commutative diagram of R-modules and R-morphisms as the following

$$\begin{array}{ccccccc} \cdots \to H^{n-1}(Hom(P''_{A''},B) & \xrightarrow{i^*} & H^{n-1}(Hom(P''_A,B)) & \xrightarrow{\sigma} & H^n(Hom(P''_{A'},B) & \longrightarrow & \ldots \\ g^*\uparrow & & h_*\uparrow & & f_*\uparrow & & \\ \cdots \to H^{n-1}(Hom(P''_{C''},B) & \xrightarrow{j^*} & H^{n-1}(Hom(P''_C,B)) & \xrightarrow{\sigma^*} & H^n(Hom(P''_{C'},B)) & \longrightarrow & \ldots \end{array}$$

and our proof will be complete by using the definition of ${}_eext^n_R(A,B) = H^n(Hom(P_A,B))$. □

2.2 Covariant essential derived functor ${}_eExt$

In this subsection we want to describe covariant right derived functor Ext^n_R on the e-injective resolution call it covariant essential derived functor (breifly ${}_eExt^R_n$) and discuss some properties of them and we prove some theorem under acceptable condition. In homology ext^R_n and Ext^R_n are equivalent but this is not the case for ${}_eext^R_n$ and ${}_eExt^R_n$. We begin with the following definition.

Definition 2.19. *An e-injective resolution of an $R-$module A is an e-exact sequence $0 \to A \overset{\eta}{\to} E^0 \overset{d^0}{\to} E^1 \overset{d^1}{\to} \cdots \to E^n \overset{d^n}{\to} E^{n+1} \to \cdots$, where each E^i is an e-injective R-module. An e-injective resolution may not be injective see [3, Example 2.3].*

Definition 2.20. *If T is a covariant functor, then*

$$(R^nT)A = H^n(TE^M) = \frac{KerTd^n}{ImTd^{n-1}},$$

where $E: 0 \to M \to E^0 \overset{d^0}{\to} E^1 \overset{d^1}{\to} \cdots$ is the e-injective resolution of an R-module M. In particular, we put $T = Hom(A,\ )$, for any R-module A we define ${}_eExt^n_R(A,\ ) = R^nT$. Then ${}_eExt^n_R(A,M) = H^n(Hom_R(A,E^M))$, which means that ${}_eExt^n_R = \frac{Kerd^n_*}{Imd^{n-1}_*}$, where $d^n_* : Hom(A,E^n) \to Hom(A,E^{n+1})$ is defined as usual by $d^n_* : f \longmapsto d^n f$.

Theorem 2.21. *Let E be an e-injective R-module. Then for any R-module A ${}_eExt^n_R(A,E) = 0$, for all $n \geq 1$.*

Proof. Since E is an e-injective module, the e-injective resolution of E is $0 \to E \xrightarrow{1_E} E \to 0$. The corresponding deleted e-injective resolution E^E is $0 \to E \to 0$. By applying $Hom(, E)$ to the deleted complex we obtain ${}_eExt^n_R(A,E) = 0$ for all $n \geq 1$. □

Corollary 2.22. *Let $0 \to A'' \to A \to A' \to 0$ be a short e-exact sequence of R-modules and P be a projective module, then there is a long e-exact sequence $0 \to Hom(P,A'') \to Hom(P,A) \to Hom(P,A') \to {}_eExt^1_R(P,A'') \to {}_eExt^1_R(P,A) \to \cdots$.*

Proof. By [3, Proposition 2.10], we have an e-exact sequence of deleted complexes $0 \to E''^{A''} \to E^A \to E'^{A'} \to 0$. If $T = Hom(P, \ )$, then $0 \to TE'^{A''} \to TE^A \to TE''^{A'} \to 0$ is still e-exact by [1, Theorem 3.1]. Then by [2, Theorem 3.2] we have an e-exact sequence $0 \to H^0(Hom(P,E'^{A''})) \to H^0(Hom(P,E^A)) \to H^0(Hom(P,E''^{A'})) \to H^1(Hom(P,E'^{A''})) \to H^1(Hom(P,E^A)) \to \cdots$. By using the definition of ${}_e$Ext and [2, Theorem 5.3] we obtain an e-exact sequence $0 \to Hom(P,A'') \to Hom(P,A) \to Hom(P,A') \to {}_eExt^1_R(P,A'') \to {}_eExt^1_R(P,A) \to \cdots$. □

Theorem 2.23. *Given a commutative diagram of R-modules having e-exact rows as the following:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A'' & \xrightarrow{i} & A & \xrightarrow{p} & A' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & C'' & \xrightarrow{j} & C & \xrightarrow{q} & C' & \longrightarrow & 0,
\end{array}$$

then there is a commutative diagram of R-modules with e-exact rows:

$$\begin{array}{ccccccc}
{}_eExt^n_R(P,A'') & \xrightarrow{i*} & {}_eExt^n_R(P,A) & \xrightarrow{p^*} & {}_eExt^n_R(P,A') & \xrightarrow{\sigma^n} & {}_eExt^{n+1}_R(P,A'') \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle f^*} \\
{}_eExt^n_R(P,C'') & \xrightarrow{j*} & {}_eExt^n_R(P,C) & \xrightarrow{q^*} & {}_eExt^n_R(P,C') & \xrightarrow{\sigma'^n} & {}_eExt^{n+1}_R(P,C'')
\end{array}$$

Proof. By [3, Proposition 2.10], we have an e-exact sequence of deleted complexes $0 \to E'^{A''} \to E^A \to E''^{A'} \to 0$. If $T = Hom(P, \ )$, then $0 \to TE'^{A''} \to TE^A \to TE''^{A'} \to 0$ is still e-exact by [1, Theorem 3.1]. By [2, Remark 3.3] there is a commutative diagram of R-modules and R-morphisms as the following

$$\begin{array}{ccccccc}
\cdots \to H^{n-1}(Hom(P,E'^A)) & \xrightarrow{p^*} & H^{n-1}(Hom(P,E^{A'})) & \xrightarrow{\sigma} & H^n(Hom(P,E^{A''})) & \longrightarrow & \ldots \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle f_*} & & \\
\cdots \to H^{n-1}(Hom(P,E'^C)) & \xrightarrow{q^*} & H^{n-1}(Hom(P,E'^{C'})) & \xrightarrow{\sigma'^*} & H^n(Hom(P,E^{C''})) & \longrightarrow & \ldots
\end{array}$$

and our proof will be complete by using the definition of ${}_eExt^n_R(P,A) = H^n(Hom(P,E^A))$. □

3 ideal transforms with respect to essential exact sequence

Throughout this section, R is assumed to be a principal ideal domain. All our work in this section applies to two particular systems of ideals; the system of ideals $\beta = (a^n)_{n\in\mathbb{N}}$ and the system of ideals $\beta = (a^n + b^n)_{n\in\mathbb{N}}$by [4, Example 3.12]. Then the ideal transforms with respect to an ideal a is defined as $D_a = \lim_{n\in\mathbb{N}} Hom(a^n, B)$. It is covariant R-linear functor as well as it is left e-exact sequence because $Hom(a^n, B)$ is a left e-exact sequence and direct limit preserves e-exactness. If a system of ideal $(a^n)_{n\in\mathbb{N}}$ is an inverse family of ideals, then there is a natural equivalence $\varinjlim_{n\in\mathbb{N}} Hom(\frac{R}{a^n}, B) \cong \Gamma_a(B)$ as well as we have an natural equivalent between $\varinjlim_{n\in\mathbb{N}} {}_eext^i_R(\frac{R}{a^n}, B) \cong {}_eH^i_a(B)$.

Theorem 3.1. *For any torsion-free R-module B. The sequence $0 \to \Gamma_a(B) \to B \to D_a(B) \to H^1_a(B) \to 0$ is an e-exact.*

Proof. The sequence $0 \to a^n \to R \to \frac{rR}{a^n} \to 0$, where $0 \neq r \in R$ is an e-exact sequence. By Corollary 2.17 induces a long e-exact sequences of ${}_eext^n_R(\,, B)$ modules since R is an e-projective R-modules and $Hom(R, B)$ is naturally isomorphic to B, then

$$0 \to Hom(\frac{rR}{a^n}, B) \to B \to Hom(a^n, B) \to {}_eext^1_R(\frac{rR}{a^n}, B) \to 0.$$

Now passing to direct limits, we get the following

$$0 \to \varinjlim_{n\in\mathbb{N}} Hom(\frac{rR}{a^n}, B) \to B \to \varinjlim_{n\in\mathbb{N}} Hom(a^n, B) \to \varinjlim_{n\in\mathbb{N}} {}_eext^1_R(\frac{rR}{a^n}, B) \to 0.$$

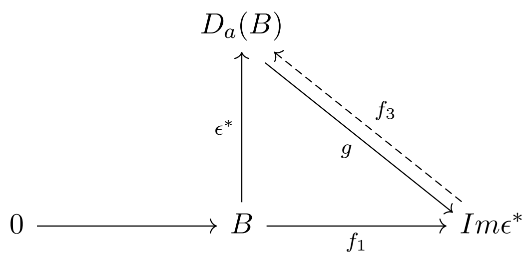
Then by using a natural equivalent induces the sequences e-exact sequence $0 \to \Gamma_a(B) \to B \to D_a(B) \to H^1_a(B) \to 0$ is an e-exact. □

Theorem 3.2. *Let a^n be a torsion-free modules. Then $Hom(a^n, B)$ is an e-injective.*

Proof. To prove that $Hom(a^n, B)$ is an e-injective, we show that $Hom(-, Hom(a^n, B))$ is an e-exact functor. By the adjoint isomorphism theorem, this functor is naturally isomorphic to $Hom(a^n \otimes -, B)$ which is the composite $Hom(-, B) \circ (a^n \otimes -)$. By [3, Proposition 2.10] $Hom(-, B)$ is an e-exact functor and by [1, Theorem 2.10] $a^n \otimes -$ is also e-exact, so their composite is again e-exact. □

Theorem 3.3. *For any torsion-free R-module B. There exists $0 \neq r \in R$ such that $\epsilon^* : B \to rD_a(B)$ is an isomorphism if and only if $\Gamma_a(B) = H^1_a(B) = 0$.*

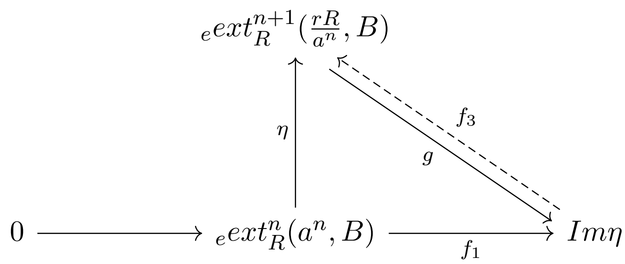
Proof. If ϵ^* is an isomorphism then by e-exactness and definition of essential we get the result. Conversely, suppose that $\Gamma_a(B) = H^1_a(B) = 0$ then by e-exactness and definition of essential we obtain $Ker\epsilon^* = 0$ implies ϵ^* is an monic. It is remain to shows that ϵ^* is an epic. By Theorem 3.2 $D_a(B)$ is an e-injective and by [3, Proposition 2.8] there exists a homomorphism $g : Im\epsilon^* \to D_a(B)$ such that $f_3 \circ g = rI_{D_a(B)}$ which means that $rD_a(B) = Im\epsilon^*$. Therefore, ϵ^* is an isomorpism.



□

Theorem 3.4. *Let ${}_e ext^n_R(a^n, B)$ be an e-injective torsion-free modules. There exists $0 \neq r \in R$ such that $\eta : {}_e ext^n_R(a^n, B) \to r_e ext^{n+1}_R(\frac{rR}{a^n}, B)$ is an isomorphism.*

Proof. By Corollary 2.17 induces a long e-exat sequences of ${}_e ext^n_R$ and by e-exactness and definition of essential we obtain $Ker\eta = 0$ implies η is an monic. It is remain to shows that η is an epic. By [3, Proposition 2.8] there exists a homomorphism $g : Im\eta \to {}_e ext^{n+1}_R(\frac{rR}{a^n}, B)$ such that $f_3 \circ g = rI_{{}_e ext^{n+1}_R(\frac{rR}{a^n}, B)}$ which means that $r_e ext^{n+1}_R(\frac{rR}{a^n}, B) = Im\eta$. Therefore, η is an isomorpism.



□

Theorem 3.5. *For any torsion-free R-module B, there is an e-exact sequences*

$$0 \to D_{r(a+b)}(B) \to D_a(B) \bigoplus D_b(B) \to D_{a\cap b}(B) \to rH^2_a D_{r(a+b)}(B)$$
$$\to rH^2_a D_a(B) \bigoplus rH^2_a D_b(B) \to rH^2_a D_{a\cap b}(B) \to \cdots .$$

Proof. we have an e-exact sequences $0 \to a \cap b \to a \bigoplus b \to \frac{r(a \bigoplus b)}{a \cap b} \to 0$, where $0 \neq r \in R$. By Corollary 2.17 induces a long e-exat sequences

$$0 \to Hom(r(a+b), B) \to Hom(a, B) \bigoplus Hom(b, B) \to Hom(a \cap b, B)$$
$$\to {}_e ext^1_R(r(a+b), B) \to {}_e ext^1_R(a, B) \bigoplus {}_e ext^1_R(a, B) \to {}_e ext^1_R(a \cap b, B)$$
$$\to \cdots .$$

Now, passing to direct limits, we obtain the following

$$
\begin{aligned}
0 \to \varinjlim_{n\in\mathbb{N}} Hom(r(a+b),B) &\to \varinjlim_{n\in\mathbb{N}} Hom(a,B) \bigoplus \varinjlim_{n\in\mathbb{N}} Hom(b,B) \\
\to \varinjlim_{n\in\mathbb{N}} Hom(a\cap b,B) &\to \varinjlim_{n\in\mathbb{N}} {}_eext^1(r(a+b),B) \to \varinjlim_{n\in\mathbb{N}} {}_eext^1_R(a,B) \\
&\bigoplus \varinjlim_{n\in\mathbb{N}} {}_eext^1_R(a,B) \to \varinjlim_{n\in\mathbb{N}} {}_eext^1_R(a\cap b,B) \to \cdots,
\end{aligned}
$$

then by using a natural equivalent and Theorem 3.4 induces an e-exact sequences

$$
\begin{aligned}
0 \to D_{r(a+b)}(B) &\to D_a(B) \bigoplus D_b(B) \to D_{a\cap b}(B) \to r_e H^2_a D_{r(a+b)}(B) \\
&\to r_e H^2_a D_a(B) \bigoplus r_e H^2_a D_b(B) \to r_e H^2_a D_{a\cap b}(B) \to \cdots .
\end{aligned}
$$

□

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